

# HOLOMORPHIC EXTENSION ON PRODUCT LIPSCHITZ SURFACES IN TWO COMPLEX VARIABLES

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**ABSTRACT.** In this work we prove a new  $L^p$  holomorphic extension result for functions defined on product Lipschitz surfaces with small Lipschitz constants in two complex variables. We define biparameter and partial Cauchy integral operators that play the role of boundary values for holomorphic functions on product Lipschitz domain. In the spirit of the application of David-Journé-Semmes and Christ's  $Tb$  theorem to the Cauchy integral operator, we prove a biparameter  $Tb$  theorem and apply it to prove  $L^p$  space bounds for the biparameter Cauchy integral operator. We also prove some new biparameter Littlewood-Paley-Stein estimates and use them to prove the biparameter  $Tb$  theorem.

## 1. INTRODUCTION

In this work, we solve a holomorphic extension problem for certain product surfaces in  $\mathbb{C}^2$  and prove some results in harmonic analysis pertaining to biparameter singular integral operators and Littlewood-Paley-Stein theory. To motivate our results, we start with a brief history of holomorphic extension and boundary values of holomorphic functions results related to our problem.

The first situation we describe is one on the upper half plane  $\mathbb{H} = \{x + it : x \in \mathbb{R}, t > 0\}$  in  $\mathbb{C}$ . Given a function  $f \in L^p(\mathbb{R})$  for  $1 < p < \infty$ , one can extend  $f$  to a holomorphic function

$$F(x + it) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(y)}{y - (x + it)} dy; \text{ for } x \in \mathbb{R}, t \neq 0.$$

This function  $F$  is a holomorphic extension of  $f$  in the sense that  $F$  is holomorphic on  $\mathbb{C} \setminus \mathbb{R}$  and  $f(x) = f_+(x) - f_-(x)$  for  $x \in \mathbb{R}$ , where

$$f_+(x) = \lim_{t \rightarrow 0^+} F(x + it) \quad \text{and} \quad f_-(x) = \lim_{t \rightarrow 0^+} F(x - it).$$

These limits hold almost everywhere in  $\mathbb{R}$  and in  $L^p(\mathbb{R})$ . Sometimes this sort of holomorphic extension result is known as a Hilbert-Riemann type problem. It also follows that  $f_{\pm} = \frac{1}{2}(\pm I + iH)f$  where  $I$  is the identity operator and  $H$  is the Hilbert transform

$$Hf(x) = \lim_{t \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \frac{x - y}{(x - y)^2 + t^2} f(y) dy.$$

There is a rich history involving the Hilbert transform and boundary behavior of holomorphic functions, which is intrinsically related to the study of Hardy spaces. The  $L^p(\mathbb{R})$  extension results mentioned here were solved by the combined work of many people in the early 1900's, including classical works of Hilbert and Riesz, among others.

The next situation we discuss is a Lipschitz perturbed upper half space of the form  $\mathbb{H}_{\Gamma} = \{\gamma(x) + it : x \in \mathbb{R}, t > 0\}$  where  $\gamma : \mathbb{R} \rightarrow \mathbb{C}$  is a Lipschitz graph. Problems related to holomorphic functions on  $\mathbb{H}_{\Gamma}$  can often be solved using the corresponding solution on  $\mathbb{H}$  and the Riemann mapping theorem, but that is not the case in general with the  $L^p$  boundary behavior of holomorphic functions on  $\mathbb{H}_{\Gamma}$ . The holomorphic extension result corresponding to the one in the

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last paragraph is the following: given a function  $g \in L^p(\Gamma)$  for  $1 < p < \infty$ , one can extend  $g$  to a holomorphic function

$$G(z+it) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\xi)}{\xi - (z+it)} d\xi; \text{ for } z \in \Gamma, t \neq 0,$$

which is a holomorphic extension of  $g$  in the sense that  $G$  is holomorphic on  $\mathbb{C} \setminus \Gamma$  and  $g(z) = g_+(z) - g_-(z)$  for  $z \in \Gamma$ , where

$$g_+(z) = \lim_{t \rightarrow 0^+} G(z+it) \quad \text{and} \quad g_-(z) = \lim_{t \rightarrow 0^+} G(z-it)$$

and these limits exist in pointwise almost everywhere on  $\Gamma$  and in  $L^p(\Gamma)$ . The boundary values of  $G$  can be realized in this setting as well by  $g_{\pm}(z) = \frac{1}{2}(\pm I + iC_{\Gamma})g(z)$ , where  $C_{\Gamma}$  is the Cauchy integral transform

$$C_{\Gamma}g(z) = \lim_{t \rightarrow 0^+} \frac{1}{\pi} \int_{\Gamma} \frac{z - \xi}{(z - \xi)^2 + t^2} g(\xi) d\xi.$$

Progressing from the extension problem on  $\mathbb{H}$  to the one on  $\mathbb{H}_{\Gamma}$  was not an easy feat. It took more than 40 years from the proof of  $L^p$  bounds for the Hilbert transform to prove the  $L^p$  bounds for the Cauchy integral transform along Lipschitz curves with small constants, which was due to Calderón [Cal77]. The proof for a general Lipschitz constant appeared some years later in works of Coifman, McIntosh and Meyer [CMM82b, CMM82a]. Later, new proofs and generalizations appeared in the work of David–Journé–Semmes [DJS85], Jones [Jon89], and Chist [Chr90], among others.

These results were extended to upper half spaces of type  $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$  in place of  $\mathbb{H}$  by Stein in terms of systems of conjugate harmonic functions, see e.g. [Ste67]. In this situation, the role of the Hilbert transform is replaced by the Riesz transforms  $R_j$  on  $\mathbb{R}^n$ , and convergence results hold in  $L^p(\mathbb{R}^n)$  for  $1 < p < \infty$  and appropriate functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  with its harmonic conjugates  $R_j f(x)$  for  $j = 1, \dots, n$ . The  $n$ -dimensional Lipschitz upper half spaces were also addressed in a series of papers, Fabes–Kenig–Neri [FKN81], Jerison–Kenig [JK82], and Kenig–Pipher [KP87]. They solved problems related to harmonic functions on upper half domains of the form  $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \{L(x) + t : x \in \mathbb{R}^n, t > 0\}$ , among others, where  $L : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Lipschitz function. In [FKN81, JK82, KP87], double layer potentials replace the Riesz transforms in Stein’s work, and their associated Hardy spaces are defined.

Another setting where this type of problem has been solved is on the product upper half plane  $\mathbb{H} \times \mathbb{H}$  in  $\mathbb{C}^2$ . The corresponding Hilbert–Riemann property for the product upper half plane is stated as follows: given a function  $f \in L^p(\mathbb{R}^2)$  for  $1 < p < \infty$ , one can extend  $f$  to a holomorphic function

$$F(x+it) = \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^2} \frac{f(y)}{(y_1 - (x_1 + it_1))(y_2 - (x_2 + it_2))} dy; \text{ for } x = (x_1, x_2) \in \mathbb{R}^2, t = (t_1, t_2)$$

with  $t_1, t_2 \neq 0$ . This function  $F$  is a holomorphic extension of  $f$  in the sense that  $F$  is holomorphic on  $(\mathbb{C} \setminus \mathbb{R}) \times (\mathbb{C} \setminus \mathbb{R})$  and  $f(x) = f_{++}(x) - f_{+-}(x) - f_{-+}(x) + f_{--}(x)$  for  $x \in \mathbb{R}^2$ , where

$$\begin{aligned} f_{++}(x) &= \lim_{t_1, t_2 \rightarrow 0^+} F(x_1 + it_1, x_2 + it_2), & f_{+-}(x) &= \lim_{t_1, t_2 \rightarrow 0^+} F(x_1 + it_1, x_2 - it_2), \\ f_{-+}(x) &= \lim_{t_1, t_2 \rightarrow 0^+} F(x_1 - it_1, x_2 + it_2), & \text{and} & \\ f_{--}(x) &= \lim_{t_1, t_2 \rightarrow 0^+} F(x_1 - it_1, x_2 - it_2). \end{aligned}$$

These limits hold almost everywhere in  $\mathbb{R}^2$  and in  $L^p(\mathbb{R}^2)$ . In this situation, it follows that  $f_{\pm, \pm} = \frac{1}{4}(\pm I + iH_1)(\pm I + iH_2)f(x)$  where  $H_1 f$  and  $H_2 f$  are the Hilbert transforms applied to the first and second variable of  $f$  respectively. These operators  $H_1$ ,  $H_2$ , and  $H_1 H_2$  are sometimes called the partial and biparameter Hilbert transforms, which are bounded on  $L^p(\mathbb{R}^2)$ , see e.g. [Fef81, FS82]. These boundedness results are related to the biparameter Hardy space theory that is addressed in [MM77, GS79, Gun80, CF80, Fef81, FS82, Fef86, Fef87], among many others. Many of these articles work on the polydisk instead of products of upper half planes, but working in these two settings is essentially equivalent; look, for example, in [GS79].

In this work, we address a holomorphic extension result similar to the ones above for product Lipschitz upper half spaces, which is stated as follows. Given an appropriate Lipschitz boundary surface  $\Gamma = \Gamma_1 \times \Gamma_2 \subset \mathbb{C}^2$  and a function  $g : \Gamma \rightarrow \mathbb{C}$ , there is a function  $G$  that is holomorphic on  $(\mathbb{C} \setminus \Gamma_1) \times (\mathbb{C} \setminus \Gamma_2)$  satisfying

$$(1.1) \quad g(z) = g_{++}(z) - g_{+-}(z) - g_{-+}(z) + g_{--}(z),$$

for  $z = (z_1, z_2) \in \Gamma$ , where

$$(1.2) \quad \begin{aligned} g_{++}(z) &= \lim_{t_1, t_2 \rightarrow 0^+} G(z_1 + it_1, z_2 + it_2), & g_{+-}(z) &= \lim_{t_1, t_2 \rightarrow 0^+} G(z_1 + it_1, z_2 - it_2), \\ g_{-+}(z) &= \lim_{t_1, t_2 \rightarrow 0^+} G(z_1 - it_1, z_2 + it_2), & \text{and} & \\ g_{--}(z) &= \lim_{t_1, t_2 \rightarrow 0^+} G(z_1 - it_1, z_2 - it_2). \end{aligned}$$

For now we leave the sense in which (1.1) holds, the sense that the limits in (1.2) hold, and the conditions on  $\Gamma$  unspecified, but these things will be defined later in this section.

Before we state our holomorphic extension result, we will set a few definitions. We say that  $G(\omega_1, \omega_2)$  is holomorphic at  $(\omega_1, \omega_2) \in \mathbb{C}^2$  if  $G$  has an absolutely convergent power series representation on a neighborhood of  $(\omega_1, \omega_2)$ . We will call the Lipschitz surfaces that we work with product Lipschitz surfaces with small Lipschitz constants, and they are defined as follows. Let  $L_1, L_2 : \mathbb{R} \rightarrow \mathbb{R}$  be Lipschitz functions. Define  $\gamma_1(x_1) = x_1 + iL_1(x_1)$ ,  $\gamma_2(x_2) = x_2 + iL_2(x_2)$ , and  $\gamma(x) = (\gamma_1(x_1), \gamma_2(x_2)) \in \mathbb{C}^2$  for  $x = (x_1, x_2) \in \mathbb{R}^2$ . Then we call  $\Gamma = \Gamma_1 \times \Gamma_2 = \gamma_1(\mathbb{R}) \times \gamma_2(\mathbb{R})$  a product Lipschitz surface in  $\mathbb{C}^2$ . We say that  $\Gamma$  is a product Lipschitz surface with small Lipschitz constants if the Lipschitz constants  $\lambda_1$  and  $\lambda_2$  of  $L_1$  and  $L_2$  respectively are both smaller than 1. The upper half space associated to  $\Gamma$  is defined  $\mathbb{H}_{\Gamma_1} \times \mathbb{H}_{\Gamma_2}$ , where  $\mathbb{H}_{\Gamma_j} = \{\gamma_j(x_j) + it_j : x_j \in \mathbb{R}, t_j > 0\}$ . We also define  $L^p(\Gamma)$  for a product Lipschitz surface  $\Gamma$  as follows. Given a product Lipschitz surface  $\Gamma = \gamma_1(\mathbb{R}) \times \gamma_2(\mathbb{R})$ , let  $L^p(\Gamma)$  be the collection of measurable functions  $g : \Gamma \rightarrow \mathbb{C}$  such that

$$\|g\|_{L^p(\Gamma)}^p = \int_{\mathbb{R}^2} |g(\gamma(x))|^p |\gamma'_1(x_1)\gamma'_2(x_2)| dx_1 dx_2 < \infty.$$

Now we state our holomorphic extension result.

**Theorem 1.** *Let  $\Gamma$  be a product Lipschitz surface with small Lipschitz constants in  $\mathbb{C}^2$  defined by  $\gamma = (\gamma_1, \gamma_2) : \mathbb{R}^2 \rightarrow \mathbb{C}^2$ . Assume that*

$$\lim_{|x_1| \rightarrow \infty} \frac{\gamma_1(x_1)}{x_1} = c_1 \quad \text{and} \quad \lim_{|x_2| \rightarrow \infty} \frac{\gamma_2(x_2)}{x_2} = c_2$$

*for some  $c_1, c_2 \in \mathbb{C}$ . If  $g \in L^p(\Gamma)$  for some  $1 < p < \infty$ , then there exists a function  $G : (\mathbb{C} \setminus \Gamma_1) \times (\mathbb{C} \setminus \Gamma_2) \rightarrow \mathbb{C}$  that is a holomorphic extension of  $g$ , where (1.1) holds almost everywhere on  $\Gamma$  and the limits in (1.2) hold in  $L^p(\Gamma)$  and pointwise almost everywhere on  $\Gamma$ .*

In addition to the problems mentioned above, some other boundary value problems related to Theorem 1 can be found in the work of Bochner [Boc44], Weinstock [Wei69], Stein [Ste70, Ste73], Jacewicz [Jac73], and Krantz [Kra80, Kra07]. These works prove a number results about the behavior of holomorphic functions on domains with smooth boundaries in  $\mathbb{C}^n$ , but the point of view taken in [Boc44, Wei69, Ste70, Ste73, Jac73, Kra80, Kra07] is different than the one taken in this work. They start with a holomorphic function  $G$  defined on a domain  $D$  and make conclusions about the  $G$  near or on the boundary  $\partial D$ . Whereas we are given a boundary  $\Gamma$  with initial data  $g$  and construct a holomorphic function  $G$  on the domain  $\mathbb{H}_{\Gamma_1} \times \mathbb{H}_{\Gamma_2}$  whose behavior at the boundary is determined by  $g$ . The meaning of boundary behavior for us is described in (1.1) and (1.2).

We take this “extension from the boundary” point of view because we want this work to emphasize the boundedness of boundary value singular integral operators that take the place of the partial and biparameter Hilbert transforms from the extension problems above; we call these operators the biparameter and partial Cauchy integral transforms, and they will be defined later in this section.

It is natural to eventually define Hardy spaces of holomorphic functions associated to our product upper half space in the same way that Hardy spaces are defined on  $\mathbb{H}$ ,  $\mathbb{H}_{\Gamma}$ ,  $\mathbb{R}_+^{n+1}$ ,  $\mathbb{R}_+^{n+1}$ , and  $\mathbb{H} \times \mathbb{H}$ . These Hardy spaces are related to the holomorphic extension problems briefly described in the beginning of the Introduction. It is also natural to expect that every holomorphic function in these new Hardy spaces would be realized as one of our extensions from the boundary  $\Gamma$ . However, we do not want to deal with the extra technicalities involved with developing these spaces in this work. Instead we focus on the holomorphic extension problem for  $\Gamma$  as stated in Theorem 1.

The situation in Theorem 1 is more general than holomorphic extension results from [Boc44, Wei69, Ste70, Ste73, Jac73, Kra80, Kra07] in terms of the regularity required for the boundary. In all of these works, the domain  $D$  is assumed to have smooth boundary, at least  $C^2$ . Whereas Theorem 1 can be viewed as a boundary result for holomorphic functions on  $\mathbb{H}_{\Gamma_1} \times \mathbb{H}_{\Gamma_2}$ , which requires only Lipschitz type smoothness for the boundary  $\Gamma$ .

To prove Theorem 1, we take an approach related to the ones in [MM77, Cha79, Fef79, GS79, Ste79, CF80], which are more geometric in nature and uses the boundedness of biparameter and partial Hilbert transforms. In place of the Hilbert transforms, we define biparameter and partial Cauchy integral transforms for  $z = (z_1, z_2) \in \Gamma$  and appropriate  $g : \Gamma \rightarrow \mathbb{C}$ ,

$$\begin{aligned} C_\Gamma g(z) &= \lim_{t_1, t_2 \rightarrow 0^+} C_t g(z); & C_t g(z) &= \frac{1}{(2\pi i)^2} \int_\Gamma \frac{z_1 - \xi_1}{(z_1 - \xi_1)^2 + t_1^2} \frac{z_2 - \xi_2}{(z_2 - \xi_2)^2 + t_2^2} g(\xi) d\xi, \\ C_\Gamma^{p1} g(z) &= \lim_{t_1, t_2 \rightarrow 0^+} C_t^{p1} g(z); & C_t^{p1} g(z) &= \frac{1}{(2\pi i)^2} \int_\Gamma \frac{z_1 - \xi_1}{(z_1 - \xi_1)^2 + t_1^2} \frac{t_2}{(z_2 - \xi_2)^2 + t_2^2} g(\xi) d\xi, \\ C_\Gamma^{p2} g(z) &= \lim_{t_1, t_2 \rightarrow 0^+} C_t^{p2} g(z); & C_t^{p2} g(z) &= \frac{1}{(2\pi i)^2} \int_\Gamma \frac{t_1}{(z_1 - \xi_1)^2 + t_1^2} \frac{z_2 - \xi_2}{(z_2 - \xi_2)^2 + t_2^2} g(\xi) d\xi. \end{aligned}$$

The limits defining  $C_\Gamma$ ,  $C_\Gamma^{p1}$ , and  $C_\Gamma^{p2}$  are taken in the following pointwise sense: given  $c \in \mathbb{C}$  and  $c_t \in \mathbb{C}$  for  $t = (t_1, t_2) \in (0, \infty)^2$ , we say  $c_t \rightarrow c$  as  $t_1, t_2 \rightarrow 0^+$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $0 < t_1, t_2 < \delta$  implies  $|c_t - c| < \varepsilon$ . We also define convergence in normed spaces as  $t_1, t_2 \rightarrow 0^+$ : given a normed function space  $X$ ,  $F \in X$ , and  $F_t \in X$  for  $t = (t_1, t_2) \in (0, \infty)^2$ , we say  $F_t \rightarrow F$  as  $t_1, t_2 \rightarrow 0^+$  if  $\|F_t - F\|_X \rightarrow 0$  as  $t_1, t_2 \rightarrow 0^+$ . The operators  $C_\Gamma g$ ,  $C_\Gamma^{p1} g$ , and  $C_\Gamma^{p2} g$  are defined initially as pointwise limits on test functions, and we will prove later that these limits hold in  $L^p(\Gamma)$  as well for  $1 < p < \infty$  and appropriate  $g$ . These convergence results will be proved in Sections 5 and 6. A crucial part of the proof of these convergence results is the  $L^p(\Gamma)$  boundedness of  $C_\Gamma$ ,  $C_\Gamma^{p1}$ , and  $C_\Gamma^{p2}$ , which we state now in Theorem 2.

**Theorem 2.** *Let  $\Gamma$  be a product Lipschitz surface with small Lipschitz constant in  $\mathbb{C}^2$  defined by  $\gamma = (\gamma_1, \gamma_2) : \mathbb{R}^2 \rightarrow \mathbb{C}^2$ . Assume that*

$$\lim_{|x_1| \rightarrow \infty} \frac{\gamma_1(x_1)}{x_1} = c_1 \quad \text{and} \quad \lim_{|x_2| \rightarrow \infty} \frac{\gamma_2(x_2)}{x_2} = c_2$$

for some  $c_1, c_2 \in \mathbb{C}$ . Then the operators  $C_\Gamma$ ,  $C_\Gamma^{p1}$ , and  $C_\Gamma^{p2}$  can be continuously extended to bounded operators on  $L^p(\Gamma)$  and for  $g \in L^p(\Gamma)$

$$\lim_{t_1, t_2 \rightarrow 0^+} C_t g = C_\Gamma g, \quad \lim_{t_1, t_2 \rightarrow 0^+} C_t^{p1} g = C_\Gamma^{p1} g, \quad \text{and} \quad \lim_{t_1, t_2 \rightarrow 0^+} C_t^{p2} g = C_\Gamma^{p2} g$$

in  $L^p(\Gamma)$  when  $1 < p < \infty$  and pointwise almost everywhere on  $\Gamma$ .

We take a moment now to discuss why Theorem 2 cannot be proved with techniques currently available in the literature; in particular why the analysis of holomorphic functions related to  $\mathbb{H} \times \mathbb{H}$  and the Hilbert transforms is not applicable to our problem. Much of the machinery used in the analysis of holomorphic functions related to  $\mathbb{H} \times \mathbb{H}$  and the Hilbert transforms is not available when we move to the setting of  $\Gamma = \Gamma_1 \times \Gamma_2$  and the Cauchy integral transforms. If one defines the biparameter Hilbert transform  $H^{bp}$  as  $C_\Gamma$  is defined above (but with  $\gamma_j(x_j) = x_j$  for  $j = 1, 2$ ), then with the aid of the Fourier transform it is easy to show that  $H^{bp} = H_1 H_2$ . Since  $H^{bp}$  can be realized as this composition of  $H_1$  and  $H_2$  in this way, the  $L^p(\mathbb{R}^2)$  bounds for  $H^{bp}$  trivially follow from those of  $H_1$  and  $H_2$ . Furthermore, the fact that  $H^{bp} = H_1 H_2$  says that the two dimensional limit  $t_1, t_2 \rightarrow 0^+$  defining  $H^{bp}$  can actually be realized as iterated one dimensional limits  $t_1 \rightarrow 0^+$  and  $t_2 \rightarrow 0^+$ . There is no such formula to write  $C_\Gamma$  as a composition  $C_\Gamma^{p1}$  and  $C_\Gamma^{p2}$  that we know of since the Fourier transform is not a viable tool in this setting; that is, it is not known in general if the two dimensional limit defining  $C_\Gamma$  can be realized as an iterated one dimensional limit. This precludes, at least with the tools currently available, any relatively simple proof of  $L^p(\Gamma)$  bounds for  $C_\Gamma$ , and hence motivates the development of the harmonic analysis theory in this article.

To obtain the pointwise convergence result stated in Theorem 2, we need a little more than boundedness of the partial and biparameter Cauchy integral transforms. For  $z = (z_1, z_2) \in \Gamma$  and appropriate functions  $g : \Gamma \rightarrow \mathbb{C}$ , we define the maximal biparameter Cauchy integral transform

$$(1.3) \quad C_\Gamma^* g(z) = \sup_{t_1, t_2 > 0} |C_t g(z)|,$$

where  $t = (t_1, t_2)$ . Then, we prove the following boundedness result.

**Theorem 3.** *Let  $\Gamma$  be as in Theorem 2. The maximal operator  $C_\Gamma^*$  extends to a bounded operator  $C_\Gamma^* : L^p(\Gamma) \rightarrow L^p(\Gamma)$  for  $1 < p < \infty$ . Moreover, for all  $g$  in  $L^p(\Gamma)$ ,  $C_\Gamma g$  converges to  $C_\Gamma g$  almost everywhere on  $\Gamma$ .*

We prove Theorem 2 using the approach that David-Journé-Semmes used to apply their  $Tb$  theorem to prove  $L^p$  bounds for Cauchy integral transform in [DJS85]. For this, we prove the following reduced biparameter  $Tb$  theorem.

**Theorem 4.** *Let  $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$  be para-accretive functions, and define  $b(x) = b_1(x_1)b_2(x_2)$  and  $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ . Also let  $T$  be a biparameter operator of Calderón-Zygmund type associated to  $b$  and  $\tilde{b}$ . If  $T$  satisfies the weak boundedness property, mixed weak boundedness properties, and the  $Tb = T^*\tilde{b} = 0$  conditions, then  $T$  can be continuously extended to a bounded linear operator on  $L^p(\mathbb{R}^n)$  for  $1 < p < \infty$ .*

There have been a number of results for biparameter singular integral operators of Calderón-Zygmund type, going back to R. Fefferman, Stein, and Journé, among others. There were different versions of  $T1$  theorems proved in R. Fefferman-Stein [FS82], Journé [Jou85], Pott-Villaroya [PV11], Han-Lin-Lee [HLL13], Ou [Ou13], and Hart-Lu-Torres [HLT]. In fact, the recent articles [HLL13, Ou13] include biparameter  $Tb$  theorems as well. The formulation of Theorem 4 is different than the ones in [HLL13, Ou13], and even the definitions of biparameter Calderón-Zygmund operators are different. In Section 4, we define biparameter singular integral operators relying only on continuity in test function spaces, a full kernel representation, and testing conditions on normalized bumps, whereas in [HLL13, Ou13] the singular integral operators addressed are required to have full and partial kernel representations as well as some a priori partial  $L^2$  bounds. In our formulation, we do not use partial kernel representations or partial  $L^2$  boundedness hypotheses. Instead, we introduce a mixed weak boundedness, which is a testing condition similar to the full weak boundedness property used in many of the aforementioned works. The formulation of Theorem 4 in this work is a natural extension of the single parameter theory, and the sufficient conditions seem to be easy to verify, as will be demonstrated in Section 5. Unfortunately, Theorem 4 is still not a full characterization of  $L^p$  bounds for biparameter Calderón-Zygmund operators since difficulties of working with product  $BMO$  persist, but this reduced  $Tb = T^*b = 0$  Theorem 4 is sufficient to prove the boundedness results in Theorem 2 and hence the holomorphic extension result of in Theorem 1. The formulation of the biparameter singular integral operators in this work is essentially the same as the one by Hart-Lu-Torres in [HLT], but we repeat the constructions to fit the accretive function setting in Theorem 4.

Even though we will only apply Theorem 4 when  $n_1 = n_2 = 1$ , we prove it for general dimensions  $n_1, n_2 \in \mathbb{N}$ . Our strategy to prove Theorem 4 is to decompose the operator  $T$ ,

$$\langle Tf, g \rangle = \sum_{\vec{k} \in \mathbb{Z}^2} \langle \Theta_{\vec{k}} f, g \rangle,$$

where  $\Theta_{\vec{k}}$  are smooth truncations of  $T$ . These truncations  $\Theta_{\vec{k}}$  are biparameter Littlewood-Paley-Stein operators, which have been studied extensively in the single parameter setting, see e.g. [DJ84, DJS85, Sem90, Han94]. There are a few results for biparameter Littlewood-Paley-Stein operators due to R. Fefferman, Stein, and Journé [Fef81, FS82, Fef86, Jou85], among others. All of these results are for operators of convolution type. We prove estimates for the square function associated to a larger class of operators including non-convolution operators, which we call biparameter Littlewood-Paley-Stein operators. In particular, we prove bounds for square function operators associated to biparameter Littlewood-Paley-Stein operators, defined by

$$(1.4) \quad Sf(x)^2 = \sum_{\vec{k} \in \mathbb{Z}} |\Theta_{\vec{k}} f(x)|^2$$

for  $x \in \mathbb{R}^n$  and appropriate  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ .

**Theorem 5.** *Let  $b_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2 \in L^\infty(\mathbb{R}^{n_2})$  be para-accretive functions, and define  $b(x) = b_1(x_1)b_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ . Also let  $\Theta_{\vec{k}}$  for  $\vec{k} \in \mathbb{Z}^2$  be a collection of biparameter Littlewood-Paley-Stein operators with kernels  $\theta_{\vec{k}}$ . If*

$$\int_{\mathbb{R}^{n_1}} \theta_{\vec{k}}(x, y) b_1(y_1) dy_1 = \int_{\mathbb{R}^{n_2}} \theta_{\vec{k}}(x, y) b_2(y_2) dy_2 = 0$$

*for all  $\vec{k} \in \mathbb{Z}^2$  and  $x, y \in \mathbb{R}^n$ , then  $\|Sf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$  for all  $f \in L^p(\mathbb{R}^n)$  when  $1 < p < \infty$ . Note that  $S$  is the square function operator defined in (1.4)*

In fact, we will prove Theorem 5 for a slightly larger class of operators than the biparameter Littlewood-Paley-Stein operators. These classes of operators will be defined in the coming sections, and it will be specified how they can be generalized to a slightly larger class by weakening the regularity properties of  $\theta_{\tilde{\kappa}}$ .

The formulations and proofs of Theorems 4 and 5 were introduced by Hart-Lu-Torres [HLT] in a slightly different setting, where  $b = \tilde{b} = 1$ . In Sections 3 and 4, we reproduce the proofs from [HLT], and address the additional technical difficulties that arise when accretive functions  $b$  and  $\tilde{b}$  are used in place of 1.

This article is organized in the following way. In Section 2, we prove the holomorphic extension result in Theorem 1 assuming that Theorem 2 holds. In Section 3, we develop some biparameter Littlewood-Paley-Stein theory and prove Theorem 5. In Section 4, we prove the biparameter  $Tb$  Theorem 4 using results from Section 3. Finally in Section 5, we prove part of Theorem 2 by applying Theorem 4 to a parameterized version of  $C_\Gamma$ , and we prove Theorem 3. In Section 6 we prove the rest of Theorem 4 by applying the one parameter  $Tb$  theorem from [DJS85] to parameterized versions of  $C_\Gamma^{p_1}$  and  $C_\Gamma^{p_2}$ .

## 2. HOLOMORPHIC EXTENSION FROM PRODUCT LIPSCHITZ DOMAINS

Fix Lipschitz functions  $L_1, L_2 : \mathbb{R} \rightarrow \mathbb{R}$  with Lipschitz constants  $\lambda_1 < 1$  and  $\lambda_2 < 1$ . Define  $\gamma_1(x_1) = x_1 + iL_1(x_1)$ ,  $\gamma_2(x_2) = x_2 + iL_2(x_2)$ , and  $\gamma(x) = (\gamma_1(x_1), \gamma_2(x_2))$  for  $x = (x_1, x_2) \in \mathbb{R}^2$ . Then  $\Gamma = \Gamma_1 \times \Gamma_2$  is a product Lipschitz surface with small Lipschitz constants in  $\mathbb{C}^2$ , where  $\Gamma_1 = \gamma_1(\mathbb{R})$  and  $\Gamma_2 = \gamma_2(\mathbb{R})$ . It follows that

$$0 < 1 - \lambda_j^2 \leq \frac{(x_j - y_j)^2 - (L_j(x_j) - L_j(y_j))^2}{(x_j - y_j)^2} = \frac{|Re[(\gamma_j(x_j) - \gamma_j(y_j))^2]|}{(x_j - y_j)^2} \leq 2.$$

Throughout this work, we will use the fact that  $Re[(\gamma_j(x_j) - \gamma_j(y_j))^2]$  and  $(x_j - y_j)^2$  are comparable with constants only depending on the Lipschitz constants of  $\gamma$ , not on  $x_j$  and  $y_j$ . We also remark that the norms of  $g$  and  $g \circ \gamma$  are comparable in the following sense: for any  $g \in L^p(\Gamma)$ ,

$$\begin{aligned} \|g \circ \gamma\|_{L^p(\mathbb{R}^2)}^p &\leq \|(\gamma_1')^{-1}\|_{L^\infty(\mathbb{R})} \|(\gamma_2')^{-1}\|_{L^\infty(\mathbb{R})} \|g\|_{L^p(\Gamma)}^p \leq \|g\|_{L^p(\Gamma)}^p \\ (2.1) \quad &\leq \|\gamma_1'\|_{L^\infty(\mathbb{R})} \|\gamma_2'\|_{L^\infty(\mathbb{R})} \|g \circ \gamma\|_{L^p(\mathbb{R}^2)}^p \leq 2 \|g \circ \gamma\|_{L^p(\mathbb{R}^2)}^p. \end{aligned}$$

Note that since  $Re[\gamma_j'(x_j)] = 1$  for all  $x_j \in \mathbb{R}$ , we have  $|\gamma_j'(x_j)| \geq Re[\gamma_j'(x_j)] = 1$  for all  $x_j \in \mathbb{R}$ . Now given a function  $g : \Gamma \rightarrow \mathbb{C}$ , we define for  $\omega = (\omega_1, \omega_2) = (z_1 + it_1, z_2 + it_2) \in (\mathbb{C} \setminus \Gamma_1) \times (\mathbb{C} \setminus \Gamma_2)$  where  $(z_1, z_2) \in \Gamma$  and  $t_1, t_2 \neq 0$ ,

$$(2.2) \quad G(\omega_1, \omega_2) = \frac{1}{(2\pi i)^2} \int_\Gamma \frac{g(\xi) d\xi}{(\xi_1 - \omega_1)(\xi_2 - \omega_2)}.$$

It follows that

$$\begin{aligned} G(\omega_1, \omega_2) &= \frac{1}{4} \int_\Gamma \left( p_{t_1}(z_1 - \xi_1) p_{t_2}(z_2 - \xi_2) - q_{t_1}(z_1 - \xi_1) q_{t_2}(z_2 - \xi_2) \right. \\ &\quad \left. + i q_{t_1}(z_1 - \xi_1) p_{t_2}(z_2 - \xi_2) + i p_{t_1}(z_1 - \xi_1) q_{t_2}(z_2 - \xi_2) \right) g(\xi) d\xi, \end{aligned}$$

where

$$p_{t_j}(\omega_j) = \frac{1}{\pi} \frac{t_j}{\omega_j^2 + t_j^2} \quad \text{and} \quad q_{t_j}(\omega_j) = \frac{1}{\pi} \frac{\omega_j}{\omega_j^2 + t_j^2} \quad \text{for } \omega_j \in \mathbb{C}.$$

Also define for  $t = (t_1, t_2) \in (0, \infty)^2$ ,  $g_1 : \Gamma_1 \rightarrow \mathbb{C}$ ,  $g_2 : \Gamma_2 \rightarrow \mathbb{C}$ ,  $g : \Gamma \rightarrow \mathbb{C}$ , and  $z = (z_1, z_2) \in \Gamma$ , the operators

$$\begin{aligned} P_{t_1} g_1(z_1) &= \int_{\Gamma_1} p_{t_1}(z_1 - \xi_1) g_1(\xi_1) d\xi_1, & P_{t_2} g_2(z_2) &= \int_{\Gamma_2} p_{t_2}(z_2 - \xi_2) g_2(\xi_2) d\xi_2, \\ \text{and } P_t g(z) &= \int_\Gamma p_{t_1}(z_1 - \xi_1) p_{t_2}(z_2 - \xi_2) g(\xi) d\xi. \end{aligned}$$

We use the indices of  $P_{t_1}$ ,  $P_{t_2}$ , and  $P_t$  to identify the operators. Note that  $P_t g = P_{t_1} P_{t_2} g$  for  $g : \Gamma \rightarrow \mathbb{C}$ , where we use the notation

$$P_{t_1} g(z) = \int_{\Gamma_1} p_{t_1}(z_1 - \xi_1) g(\xi_1, z_2) d\xi_1 \quad \text{and} \quad P_{t_2} g(z) = \int_{\Gamma_2} p_{t_2}(z_2 - \xi_2) g(z_1, \xi_2) d\xi_2$$

This is an abuse of notation, but it is clear in context which operator is being used. We start with a lemma about the convergence of the operators  $P_{t_1}g$ ,  $P_{t_2}g$ , and  $P_tg$  for  $g \in L^p(\Gamma)$ .

**Lemma 2.1.** *Let  $\Gamma$  be a product Lipschitz surface with small Lipschitz constants in  $\mathbb{C}^2$  and  $g \in L^p(\Gamma)$  for some  $1 < p < \infty$ . Then*

$$\lim_{t_1 \rightarrow 0^+} P_{t_1}g = g, \quad \lim_{t_2 \rightarrow 0^+} P_{t_2}g = g, \quad \text{and} \quad \lim_{t_1, t_2 \rightarrow 0^+} P_tg = g,$$

where each limit holds in the topology of  $L^p(\Gamma)$  and pointwise almost everywhere on  $\Gamma$ .

*Proof.* We first verify that  $P_{t_j}1 = 1$  for each  $j = 1, 2$ . Let  $R > 0$  and

$$E_R = \{z_j \in \Gamma_j : |z_j| \leq R\} \cup \{z_j \in \mathbb{C} : |z_j| = R, \operatorname{Im}(z_j) > L_j(\operatorname{Re}(z_j))\}.$$

$E_R$  is a closed, and for  $R$  sufficiently large, it defines the boundary of an open, simply connected region  $U_R = \{z_j \in \mathbb{C} : |z_j| < R, \operatorname{Im}(z_j) > L_j(\operatorname{Re}(z_j))\}$ . For  $z_j \in \Gamma_j, t_j > 0$ , and  $R$  sufficiently large, it follows that  $z_j + it_j \in U_R$  and  $z_j - it_j \notin U_R$ . Then

$$\frac{t_j}{\xi_j - (z_j - it_j)}$$

is holomorphic in  $\xi_j$  on  $U_R$  for such  $z_j, t_j$ , and  $R$ . Using the decay of  $p_{t_j}$  and a residue theorem, it follows that

$$\begin{aligned} \int_{\Gamma_j} p_{t_j}(z_j - \xi_j) d\xi_j &= \lim_{R \rightarrow \infty} \frac{1}{\pi} \int_{E_R} \frac{t_j}{(\xi_j - (z_j + it_j))(\xi_j - (z_j - it_j))} d\xi_j \\ &= \lim_{R \rightarrow \infty} \frac{1}{\pi} \frac{2\pi i t_j}{(z_j + it_j) - (z_j - it_j)} = 1. \end{aligned}$$

Consider the following parameterized versions of  $P_t$ ,  $P_{t_1}$ , and  $P_{t_2}$ : for  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  and  $x \in \mathbb{R}^2$

$$\tilde{P}_{t_1}f(x) = \int_{\mathbb{R}} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) \gamma'_1(y_1) f(y_1, x_2) dy_1,$$

$$\tilde{P}_{t_2}f(x) = \int_{\mathbb{R}} p_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) \gamma'_2(y_2) f(x_1, y_2) dy_2, \text{ and}$$

$$\tilde{P}_t f(x) = \tilde{P}_{t_1} \tilde{P}_{t_2} f(x) = \int_{\mathbb{R}^2} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) p_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) \gamma'_1(y_1) \gamma'_2(y_2) f(y) dy.$$

The kernels of  $\tilde{P}_{t_1}$ ,  $\tilde{P}_{t_2}$ , and  $\tilde{P}_t$  are

$$\begin{aligned} \tilde{p}_{t_1}(x_1, y_1) &= p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) \gamma'_1(y_1), & \tilde{p}_{t_2}(x_2, y_2) &= p_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) \gamma'_2(y_2), \\ \text{and } \tilde{p}_t(x, y) &= \tilde{p}_{t_1}(x_1, y_1) \tilde{p}_{t_2}(x_2, y_2), & \text{respectively.} \end{aligned}$$

Note that  $\tilde{P}_{t_j}1(x_j) = P_{t_j}1(\gamma_j(x_j)) = 1$  for all  $x_j \in \mathbb{R}$ . Also since the Lipschitz constant of  $L_1$  and  $L_2$  are small, it follows that

$$|\tilde{p}_{t_j}(x_j, y_j)| = \frac{1}{\pi} \left| \frac{t_j |\gamma'_j(y_j)|}{t_j^2 + (\gamma_j(x_j) - \gamma_j(y_j))^2} \right| \leq \frac{t_j}{t_j^2 + (1 - \lambda_j^2)(x_j - y_j)^2} \lesssim \frac{t_j^{-1}}{(1 + t_j^{-1}|x_j - y_j|)^2}.$$

Then  $\{\tilde{p}_{t_j} : t_j > 0\}$  forms an approximation to identity on  $\mathbb{R}$  for each  $j = 1, 2$ . Fix  $g \in L^p(\Gamma)$  for some  $1 < p < \infty$ . It follows that  $g \circ \gamma \in L^p(\mathbb{R}^2)$ , and hence that  $g \circ \gamma(\cdot, x_2) \in L^p(\mathbb{R})$  for almost every  $x_2 \in \mathbb{R}$ . Now fix  $x_2 \in \mathbb{R}$  outside of an appropriate exceptional set, so that  $\|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})} < \infty$ . It follows that  $g \circ \gamma(\cdot, x_2) \in L^p(\mathbb{R})$  and hence that

$$\lim_{t_1 \rightarrow 0^+} \|\tilde{P}_{t_1}(g \circ \gamma)(\cdot, x_2) - g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})} = 0.$$

By dominated convergence, it also follows that

$$\lim_{t_1 \rightarrow 0^+} \|\tilde{P}_{t_1}(g \circ \gamma) - g \circ \gamma\|_{L^p(\mathbb{R}^2)}^p = \int_{\mathbb{R}} \lim_{t_1 \rightarrow 0^+} \|\tilde{P}_{t_1}(g \circ \gamma)(\cdot, x_2) - g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})}^p dx_2 = 0.$$

Therefore  $\tilde{P}_{t_1}(g \circ \gamma) \rightarrow g \circ \gamma$  in  $L^p(\mathbb{R}^2)$ , and in light of (2.1) it easily follows that  $P_{t_1}g \rightarrow g$  in  $L^p(\Gamma)$ . By symmetry, it follows that  $P_{t_2}g \rightarrow g$  in  $L^p(\Gamma)$  as well. Now for  $g \in L^p(\Gamma)$ , we verify that  $P_tg \rightarrow g$  in  $L^p(\Gamma)$  as  $t_1, t_2 \rightarrow 0^+$  for

$1 < p < \infty$ , as defined in the introduction. First, define  $\mathcal{M}_1$  to be the Hardy-Littlewood maximal function acting on the first variable of a function  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ , i.e.

$$\mathcal{M}_1 f(x) = \sup_{I \ni x_1} \frac{1}{|I|} \int_I |f(y_1, x_2)| dy_1,$$

where the supremum is taken over all intervals  $I \subset \mathbb{R}$  that contain  $x_1$ . It is not hard to verify that  $\mathcal{M}_1$  is bounded on  $L^p(\mathbb{R}^2)$  for  $1 < p \leq \infty$  and that  $|P_{t_1} h(\gamma(x))| \lesssim \mathcal{M}_1(h \circ \gamma)(x)$  uniformly in  $t_1 > 0$  for any  $h \in L^p(\Gamma)$ . The  $L^p(\Gamma)$  convergence of  $P_t g$  follows:

$$\begin{aligned} \lim_{t_1, t_2 \rightarrow 0} \|P_t g - g\|_{L^p(\Gamma)} &\leq \lim_{t_1, t_2 \rightarrow 0} \|P_{t_1}(P_{t_2} g - g)\|_{L^p(\Gamma)} + \|P_{t_1} g - g\|_{L^p(\Gamma)} \\ &\lesssim \lim_{t_1, t_2 \rightarrow 0} \|\mathcal{M}_1(\tilde{P}_{t_2}(g \circ \gamma) - g \circ \gamma)\|_{L^p(\mathbb{R}^2)} + \|P_{t_1} g - g\|_{L^p(\Gamma)} \\ &\lesssim \lim_{t_2 \rightarrow 0} \|\tilde{P}_{t_2}(g \circ \gamma) - g \circ \gamma\|_{L^p(\mathbb{R}^2)} + \lim_{t_1 \rightarrow 0} \|P_{t_1} g - g\|_{L^p(\Gamma)} = 0. \end{aligned}$$

In the last line, we use that  $\tilde{P}_{t_2}(g \circ \gamma) \rightarrow g \circ \gamma$  in  $L^p(\mathbb{R}^2)$  and that  $P_{t_1}(g \circ \gamma) \rightarrow g \circ \gamma$  in  $L^p(\mathbb{R}^2)$ . This completes the proof of the  $L^p(\Gamma)$  convergence properties in Lemma 2.1. Now we prove the pointwise convergence results. For  $g \in L^p(\Gamma)$ , it follows that  $g \circ \gamma(\cdot, x_2) \in L^p(\mathbb{R})$  for almost every  $x_2 \in \mathbb{R}$ . For a fixed  $x_2 \in \mathbb{R}$  outside of an appropriate measure zero set, by the Lebesgue differentiation theorem it follows that

$$\lim_{t_1 \rightarrow 0^+} \tilde{P}_{t_1}(g \circ \gamma)(x_1, x_2) = g(\gamma(x_1, x_2))$$

for almost every  $x_1 \in \mathbb{R}$ . Hence  $\tilde{P}_{t_1}(g \circ \gamma) \rightarrow g \circ \gamma$  as  $t_1 \rightarrow 0^+$  pointwise almost everywhere in  $\mathbb{R}^2$  and hence that  $P_{t_1} g \rightarrow g$  as  $t_1 \rightarrow 0^+$  pointwise almost everywhere in  $\Gamma$ . By symmetry,  $\tilde{P}_{t_2}(g \circ \gamma) \rightarrow g \circ \gamma$  as  $t_2 \rightarrow 0^+$  pointwise almost everywhere in  $\mathbb{R}^2$  and hence that  $P_{t_2} g \rightarrow g$  as  $t_2 \rightarrow 0^+$  pointwise almost everywhere in  $\Gamma$ .

Now we verify the pointwise convergence for  $P_t g$  on  $\Gamma$ . Fix  $x \in \mathbb{R}^2$  such that  $\tilde{P}_{t_1}(g \circ \gamma)(x) \rightarrow g \circ \gamma(x)$  as  $t_1 \rightarrow 0^+$  and  $\|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})} < \infty$ , which is true for almost every  $x \in \mathbb{R}^2$ . Now we bound

$$\begin{aligned} |\tilde{P}_t(g \circ \gamma)(x) - g \circ \gamma(x)| &\leq |\tilde{P}_{t_1}(\tilde{P}_{t_2}(g \circ \gamma) - (g \circ \gamma))(x)| + |\tilde{P}_{t_1}(g \circ \gamma)(x) - (g \circ \gamma)(x)| \\ (2.3) \quad &\lesssim \int_{\mathbb{R}} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) |\tilde{P}_{t_2}(g \circ \gamma)(y_1, x_2) - (g \circ \gamma)(y_1, x_2)| dy_1 \\ &\quad + |\tilde{P}_{t_1}(g \circ \gamma)(x) - (g \circ \gamma)(x)|. \end{aligned}$$

We verify that the first term of (2.3) tends to zero as  $t_1, t_2 \rightarrow 0^+$ : let  $\varepsilon > 0$ . Since  $\tilde{P}_{t_2}(g \circ \gamma)(y_1, x_2) \rightarrow (g \circ \gamma)(y_1, x_2)$  pointwise as  $t_2 \rightarrow 0^+$  for almost every  $y_1 \in \mathbb{R}$ , there exists  $\delta > 0$  such that  $0 < t_2 < \delta$  implies  $|\tilde{P}_{t_2}(g \circ \gamma)(y_1, x_2) - (g \circ \gamma)(y_1, x_2)| < \varepsilon$  for almost every  $y_1 \in \mathbb{R}$  such that  $|x_1 - y_1| \leq 1$  (recall we have fixed  $x_1$  and  $x_2$ ). The selection of  $\delta$  does not depend on  $y_1$  as long as it is within the compact set defined by  $|x_1 - y_1| \leq 1$ . Now we take  $0 < t_1, t_2 < \min(\delta, \varepsilon)/(1 + \|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})})$ , which is possible since  $x \in \mathbb{R}^2$  was selected so that  $\|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})}$  is finite. Then

$$\begin{aligned} &\int_{\mathbb{R}} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) |\tilde{P}_{t_2}(g \circ \gamma)(y_1, x_2) - (g \circ \gamma)(y_1, x_2)| dy_1 \\ &\lesssim \varepsilon \int_{|x_1 - y_1| \leq 1} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) dy_1 \\ &\quad + \int_{|x_1 - y_1| > 1} \frac{t_1(|\tilde{P}_{t_2}(g \circ \gamma)(y_1, x_2)| + |g \circ \gamma(y_1, x_2)|)}{(\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2} dy_1 \\ &\lesssim \varepsilon + t_1 \int_{|x_1 - y_1| > 1} \frac{(|\tilde{P}_{t_2}(g \circ \gamma)(y_1, x_2)| + |g \circ \gamma(y_1, x_2)|)}{(x_1 - y_1)^2} dy_1 \\ &\lesssim \varepsilon + t_1 \left( \|\tilde{P}_{t_2}(g \circ \gamma)(\cdot, x_2)\|_{L^p(\mathbb{R})} + \|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})} \right) \left( \int_{|x_1 - y_1| > 1} \frac{dy_1}{(x_1 - y_1)^{2p'}} \right)^{\frac{1}{p'}} \\ &\lesssim \varepsilon + t_1 \|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})} \lesssim \varepsilon. \end{aligned}$$



It follows that the first term of (2.3) tends to zero as  $t_1, t_2 \rightarrow 0^+$  for almost every  $x \in \mathbb{R}^2$ . The second term in (2.3) also tends to zero as  $t_1, t_2 \rightarrow 0^+$  since  $x$  was chosen so that  $\tilde{P}_{t_1} f(x) \rightarrow f(x)$  as  $t_1 \rightarrow 0^+$ . Again using (2.1), it easily follow that  $P_t g \rightarrow g$  as  $t_1, t_2 \rightarrow 0^+$  pointwise almost everywhere on  $\Gamma$ .  $\square$

Now we prove Theorem 1 assuming Theorem 2; we will prove Theorem 2 in Section 5.

*Proof.* Let  $1 < p < \infty$ ,  $g \in L^p(\Gamma)$ , and define  $G$  as in (2.2). Note that  $p_{-t_j}(z_j - \xi_j) = -p_{t_j}(z_j - \xi_j)$  and  $q_{-t_j}(z_j - \xi_j) = q_{t_j}(z_j - \xi_j)$  for  $t_j \neq 0$ ,  $z_j \in \Gamma_j$ , and  $j = 1, 2$ . Then it follows that for  $(z_1, z_2) \in \Gamma$  and  $t_1, t_2 > 0$ , we have

$$\begin{aligned} G(z_1 + it_1, z_2 + it_2) &= \frac{1}{4} \left( P_t g(z) - C_t g(z) + iC_t^{p_1} g(z) + iC_t^{p_2} g(z) \right), \\ G(z_1 + it_1, z_2 - it_2) &= \frac{1}{4} \left( -P_t g(z) - C_t g(z) - iC_t^{p_1} g(z) + iC_t^{p_2} g(z) \right), \\ G(z_1 - it_1, z_2 + it_2) &= \frac{1}{4} \left( -P_t g(z) - C_t g(z) + iC_t^{p_1} g(z) - iC_t^{p_2} g(z) \right), \\ G(z_1 - it_1, z_2 - it_2) &= \frac{1}{4} \left( P_t g(z) - C_t g(z) - iC_t^{p_1} g(z) - iC_t^{p_2} g(z) \right). \end{aligned}$$

By Theorem 2, it follows that  $C_\Gamma g, C_\Gamma^{p_1} g, C_\Gamma^{p_2} g \in L^p(\Gamma)$  and  $C_t g \rightarrow C_\Gamma g$ ,  $C_t^{p_1} g \rightarrow C_\Gamma^{p_1} g$ , and  $C_t^{p_2} g \rightarrow C_\Gamma^{p_2} g$  as  $t_1, t_2 \rightarrow 0^+$  in  $L^p(\Gamma)$  and pointwise almost everywhere on  $\Gamma$ . Then for  $z = (z_1, z_2) \in \Gamma$

$$\begin{aligned} g_{++}(z) &= \frac{1}{4} \left( g(z) - C_\Gamma g(z) + iC_\Gamma^{p_1} g(z) + iC_\Gamma^{p_2} g(z) \right), \\ g_{+-}(z) &= \frac{1}{4} \left( -g(z) - C_\Gamma g(z) - iC_\Gamma^{p_1} g(z) + iC_\Gamma^{p_2} g(z) \right), \\ g_{-+}(z) &= \frac{1}{4} \left( -g(z) - C_\Gamma g(z) + iC_\Gamma^{p_1} g(z) - iC_\Gamma^{p_2} g(z) \right), \text{ and} \\ g_{--}(z) &= \frac{1}{4} \left( g(z) - C_\Gamma g(z) - iC_\Gamma^{p_1} g(z) - iC_\Gamma^{p_2} g(z) \right). \end{aligned}$$

Then it also follows that (1.1) holds, i. e.  $g = g_{++} - g_{+-} - g_{-+} + g_{--}$ , as  $L^p(\Gamma)$  functions and almost everywhere in  $\Gamma$ . It is also not hard to verify that  $G(\omega_1, \omega_2)$  is holomorphic for  $(\omega_1, \omega_2) \in (\mathbb{C} \setminus \Gamma_1) \times (\mathbb{C} \setminus \Gamma_2)$ : for  $\zeta = (\zeta_1, \zeta_2) \in (\mathbb{C} \setminus \Gamma_1) \times (\mathbb{C} \setminus \Gamma_2)$ , we have the following power series representation

$$G(\omega_1, \omega_2) = \frac{1}{(2\pi i)^2} \sum_{k_1, k_2=0}^{\infty} \left( \int_{\Gamma} \frac{g(\xi) d\xi}{(\xi_1 - \zeta_1)^{k_1+1} (\xi_2 - \zeta_2)^{k_2+1}} \right) (\omega_1 - \zeta_1)^{k_1} (\omega_2 - \zeta_2)^{k_2},$$

when  $|\omega_1 - \zeta_1| < \text{dist}(\zeta_1, \Gamma_1)/2$  and  $|\omega_2 - \zeta_2| < \text{dist}(\zeta_2, \Gamma_2)/2$ . Therefore  $G$  is a holomorphic extension of  $g$ .  $\square$

### 3. LITTLEWOOD-PALEY SQUARE FUNCTION THEORY

In this section, we develop some biparameter Littlewood-Paley-Stein theory. We work in arbitrary dimension  $\mathbb{R}^n$ , where  $n = n_1 + n_2$ . We start by fixing some notation and defining biparameter Littlewood-Paley-Stein operators and square function. For  $k_j \in \mathbb{Z}$ ,  $N_j > 0$ , and  $x_j \in \mathbb{R}$

$$\Phi_{k_j}^{N_j}(x_j) = \frac{2^{n_j k_j}}{(1 + 2^{k_j} |x_j|)^{N_j}}$$

for  $j = 1, 2$ . Again we will use the subscripts of  $k_j$ ,  $N_j$ , and  $x_j$  to distinguish between functions on  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2}$ . A collection of functions  $\theta_{\vec{k}} : \mathbb{R}^{2n} \rightarrow \mathbb{C}$  for  $\vec{k} \in \mathbb{Z}^2$  is a collection of biparameter Littlewood-Paley-Stein kernels if for all

$x_1, y_1, x'_1, y'_1 \in \mathbb{R}^{n_1}$  and  $x_2, y_2, x'_2, y'_2 \in \mathbb{R}^{n_2}$

$$(3.1) \quad |\theta_{\vec{k}}(x, y)| \lesssim \Phi_{k_1}^{N_1+\gamma}(x_1 - y_1) \Phi_{k_2}^{N_2+\gamma}(x_2 - y_2)$$

$$(3.2) \quad |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x'_1, x_2, y)| \lesssim (2^{k_1} |x_1 - x'_1|)^\gamma \times \left( \Phi_{k_1}^{N_1+\gamma}(x_1 - y_1) + \Phi_{k_1}^{N_1+\gamma}(x'_1 - y_1) \right) \Phi_{k_2}^{N_2}(x_2 - y_2)$$

$$(3.3) \quad |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x_1, x'_2, y)| \lesssim (2^{k_2} |x_2 - x'_2|)^\gamma \times \Phi_{k_1}^{N_1}(x_1 - y_1) \left( \Phi_{k_2}^{N_2+\gamma}(x_2 - y_2) + \Phi_{k_2}^{N_2+\gamma}(x'_2 - y_2) \right)$$

$$(3.4) \quad |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x, y'_1, y_2)| \lesssim (2^{k_1} |y_1 - y'_1|)^\gamma \times \left( \Phi_{k_1}^{N_1+\gamma}(x_1 - y_1) + \Phi_{k_1}^{N_1+\gamma}(x_1 - y'_1) \right) \Phi_{k_2}^{N_2}(x_2 - y_2)$$

$$(3.5) \quad |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x, y_1, y'_2)| \lesssim (2^{k_2} |y_2 - y'_2|)^\gamma \times \Phi_{k_1}^{N_1}(x_1 - y_1) \left( \Phi_{k_2}^{N_2+\gamma}(x_2 - y_2) + \Phi_{k_2}^{N_2+\gamma}(x_2 - y'_2) \right)$$

for some  $N_1 > n_1$ ,  $N_2 > n_2$ , and  $0 < \gamma \leq 1$ . We say that a collection of operators  $\Theta_{\vec{k}}$  for  $\vec{k} \in \mathbb{Z}^2$  is a collection of biparameter Littlewood-Paley-Stein operators if

$$(3.6) \quad \Theta_{\vec{k}} f(x) = \int_{\mathbb{R}^n} \theta_{\vec{k}}(x, y) f(y) dy.$$

for some collection of biparameter Littlewood-Paley-Stein kernels  $\theta_{\vec{k}}$  satisfying (3.1)-(3.5).

*Remark 3.1.* Properties (3.1)-(3.5) hold if and only if  $\theta_{\vec{k}}$  satisfies the alternate condition set:

$$\begin{aligned} |\theta_{\vec{k}}(x, y)| &\lesssim \Phi_{k_1}^{N'_1}(x_1 - y_1) \Phi_{k_2}^{N'_2}(x_2 - y_2), \\ |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x'_1, x_2, y)| &\lesssim 2^{n_1 k_1} 2^{n_2 k_2} (2^{k_1} |x_1 - x'_1|)^\gamma, \\ |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x_1, x'_2, y)| &\lesssim 2^{n_1 k_1} 2^{n_2 k_2} (2^{k_2} |x_2 - x'_2|)^\gamma, \\ |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x, y'_1, y_2)| &\lesssim 2^{n_1 k_1} 2^{n_2 k_2} (2^{k_1} |y_1 - y'_1|)^\gamma, \\ |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x, y_1, y'_2)| &\lesssim 2^{n_1 k_1} 2^{n_2 k_2} (2^{k_2} |y_2 - y'_2|)^\gamma \end{aligned}$$

for some  $N'_1 > n_1$ ,  $N'_2 > n_2$ , and  $0 < \gamma' \leq 1$ .

*Proof.* It is obvious that (3.1)-(3.5) imply the above condition set since  $\Phi_{k_j}^{N_j}(x_j) \leq 2^{k_j n_j}$ . Assume there exist  $N'_1 > n_1$ ,  $N'_2 > n_2$ , and  $0 < \gamma' \leq 1$  such that the alternate condition set holds and choose  $\eta \in (0, 1)$  small enough so that  $N_1 = (1 - \eta)N'_1 - \eta\gamma' > n_1$  and  $N_2 = (1 - \eta)N'_2 - \eta\gamma' > n_2$ , which is possible since  $N'_1 > n_1$  and  $N'_2 > n_2$ . Also define  $\gamma = \eta\gamma'$ , and it follows that

$$\begin{aligned} |\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x'_1, x_2, y)| &\lesssim \left( 2^{k_1 n_1} 2^{k_2 n_2} (2^{k_1} |x_1 - x'_1|)^\gamma \right)^\eta \\ &\quad \times \left( \Phi_{k_1}^{N'_1}(x_1 - y_1) + \Phi_{k_1}^{N'_1}(x'_1 - y_1) \right)^{1-\eta} \Phi_{k_2}^{N'_2}(x_2 - y_2)^{1-\eta} \\ &\lesssim (2^{k_1} |x_1 - x'_1|)^\gamma \left( \Phi_{k_1}^{N_1+\gamma}(x_1 - y_1) + \Phi_{k_1}^{N_1+\gamma}(x'_1 - y_1) \right) \Phi_{k_2}^{N_2+\gamma}(x_2 - y_2). \end{aligned}$$

The other conditions follow by symmetry, and hence the condition sets are equivalent.  $\square$

We use the definition of para-accretive given by Han in [Han94].

**Definition 3.1.** A function  $b \in L^\infty(\mathbb{R}^n)$  is para-accretive if  $b^{-1} \in L^\infty(\mathbb{R}^n)$  and there exists a  $c_0 > 0$  such that for all cubes  $Q \subset \mathbb{R}^n$  there exists a cube  $R \subset Q$  such that

$$\frac{1}{|Q|} \left| \int_R b(x) dx \right| \geq c_0.$$

Let  $\varphi \in C_0^\infty(\mathbb{R}^n)$  be non-negative with integral 1 and  $\text{supp}(\varphi) \subset B(0, 1/8)$ . Define for  $x \in \mathbb{R}^n$ ,  $k \in \mathbb{Z}$ , and  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ ,  $P_k f(x) = \varphi_k * f(x)$  where  $\varphi_k(x) = 2^{kn} \varphi(2^k x)$  and

$$(3.7) \quad S_k^b f(x) = P_k M_{(P_k b)^{-1}} P_k f(x) \quad \text{and} \quad D_k^b f(x) = S_{k+1}^b f(x) - S_k^b f(x).$$

Here  $M_b$  is the pointwise multiplication operator defined by  $M_b f(x) = b(x) f(x)$ . These operators were introduced by David-Journé-Semmes in [DJS85], and in that work it was proved that  $|P_k b(x)| \geq C c_0$  where the constant  $C > 0$  depends only on the dimension  $n$ . It also follows that

$$(3.8) \quad \lim_{k \rightarrow \infty} S_k^b M_b f = f \quad \text{and} \quad \lim_{k \rightarrow \infty} S_{-k}^b M_b f = 0$$

in  $L^p(\mathbb{R}^n)$  for all  $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$  when  $1 < q < p < \infty$ . We also have the following properties for  $S_k^b$  and  $D_k^b$  and their kernels  $s_k^b$  and  $d_k^b$ , see [DJS85] or [Han94] for details:

$$\begin{aligned} s_k^b(x, y) &= d_k^b(x, y) = 0 \quad \text{for } 2^k |x - y| > 1, \\ |s_k^b(x, y)| + |d_k^b(x, y)| &\lesssim 2^{kn}, \\ |s_k^b(x, y) - s_k^b(x', y)| + |d_k^b(x, y) - d_k^b(x', y)| &\lesssim 2^{kn} (2^k |x - x'|)^\gamma, \\ |s_k^b(x, y) - s_k^b(x, y')| + |d_k^b(x, y) - d_k^b(x, y')| &\lesssim 2^{kn} (2^k |y - y'|)^\gamma. \end{aligned}$$

Also let  $\mathcal{M}_S$  be the biparameter strong maximal function

$$\mathcal{M}_S f(x) = \sup_{Q_1 \ni x_1} \frac{1}{|Q_1| |Q_2|} \int_{Q_1 \times Q_2} |f(y_1, y_2)| dy_1 dy_2$$

where the supremum is taken over cubes  $Q_1 \subset \mathbb{R}^{n_1}$  and  $Q_2 \subset \mathbb{R}^{n_2}$ . It follows by standard arguments that for all  $f \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$

$$(3.9) \quad \sup_{k_1, k_2 \in \mathbb{Z}} (\Phi_{k_1}^{N_1} \otimes \Phi_{k_2}^{N_2}) * |f|(x) \lesssim \mathcal{M}_S f(x)$$

for any  $N_1 > n_1$  and  $N_2 > n_2$ . We now prove an almost orthogonality lemma.

**Lemma 3.1.** *Assume that  $\Theta_{\vec{k}}$  and  $\Psi_{\vec{k}}$  are operators defined by (3.6) with kernels respectively  $\theta_{\vec{k}}$  and  $\psi_{\vec{k}}$ . Also assume that  $\theta_{\vec{k}}$  satisfies (3.1), (3.4), and (3.5) and that  $\psi_{\vec{k}}$  satisfies (3.1), (3.2), and (3.3). If there exist para-accretive functions  $b_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2 \in L^\infty(\mathbb{R}^{n_2})$  such that*

$$\int_{\mathbb{R}^{n_j}} \theta_{\vec{k}}(x, y) b_j(y_j) dy_j = \int_{\mathbb{R}^{n_j}} \psi_{\vec{k}}(x, y) b_j(y_j) dy_j = 0$$

for  $j = 1, 2$  all  $x \in \mathbb{R}^n$  and  $k_1, k_2 \in \mathbb{Z}$ , then for all  $\vec{k} = (k_1, k_2)$ ,  $\vec{j} = (j_1, j_2) \in \mathbb{Z}^2$

$$|\Theta_{\vec{k}} M_b \Psi_{\vec{j}} f(x)| \lesssim 2^{-\varepsilon |j_1 - k_1|} 2^{-\varepsilon |j_2 - k_2|} \mathcal{M}_S f(x)$$

for some  $\varepsilon > 0$ , where  $b(x) = b_1(x_1) b_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^n$ .

*Proof.* Using the cancellation of  $\psi_{\vec{j}}$  and conditions (3.1) and (3.4), it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \theta_{\vec{k}}(x, u) b(u) \psi_{\vec{j}}(u, y) du \right| &\lesssim \int_{\mathbb{R}^n} |\theta_{\vec{k}}(x, u) - \theta_{\vec{k}}(x, y_1, u_2)| |\psi_{\vec{j}}(u, y)| du \\ &\lesssim \int_{\mathbb{R}^n} (2^{k_1} |u_1 - y_1|)^\gamma \left( \Phi_{k_1}^{N_1+\gamma}(x_1 - u_1) + \Phi_{k_1}^{N_1+\gamma}(x_1 - y_1) \right) \Phi_{k_2}^{N_2+\gamma}(x_2 - u_2) \Phi_{j_1}^{N_1+\gamma}(u_1 - y_1) \Phi_{j_2}^{N_2+\gamma}(u_2 - y_2) du \\ &= 2^{\gamma(k_1 - j_1)} \int_{\mathbb{R}^n} (2^{j_1} |u_1 - y_1|)^\gamma \Phi_{j_1}^{N_1+\gamma}(u_1 - y_1) \left( \Phi_{k_1}^{N_1+\gamma}(x_1 - u_1) + \Phi_{k_1}^{N_1+\gamma}(x_1 - y_1) \right) \\ &\quad \times \Phi_{k_2}^{N_2+\gamma}(x_2 - u_2) \Phi_{j_2}^{N_2+\gamma}(u_2 - y_2) du \\ &\leq 2^{\gamma(k_1 - j_1)} \int_{\mathbb{R}^n} \Phi_{j_1}^{N_1}(u_1 - y_1) \left( \Phi_{k_1}^{N_1+\gamma}(x_1 - u_1) + \Phi_{k_1}^{N_1+\gamma}(x_1 - y_1) \right) du_1 \int_{\mathbb{R}^n} \Phi_{k_2}^{N_2+\gamma}(x_2 - u_2) \Phi_{j_2}^{N_2+\gamma}(u_2 - y_2) du_2 \\ &\lesssim 2^{\gamma(k_1 - j_1)} \left( \Phi_{k_1}^{N_1}(x_1 - y_1) + \Phi_{j_1}^{N_1}(x_1 - y_1) \right) \left( \Phi_{k_2}^{N_2}(x_2 - y_2) + \Phi_{j_2}^{N_2}(x_2 - y_2) \right). \end{aligned}$$

By similar computations using the cancellation of  $\Theta_{\vec{k}}$ , we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \Theta_{\vec{k}}(x, u) b(u) \Psi_{\vec{j}}(u, y) du \right| \\ & \lesssim 2^{-\gamma(j_1 - k_1)} \left( \Phi_{k_1}^{N_1}(x_1 - y_1) + \Phi_{j_1}^{N_1}(x_1 - y_1) \right) \left( \Phi_{k_2}^{N_2}(x_2 - y_2) + \Phi_{j_2}^{N_2}(x_2 - y_2) \right). \end{aligned}$$

Then it follows that

$$|\Theta_{\vec{k}} M_b \Psi_{\vec{j}} f(x)| \lesssim 2^{-\gamma|j_1 - k_1|} \mathcal{M}_S f(x).$$

Our assumptions are symmetric in  $k_1, j_1$  and  $k_2, j_2$ , so it follows that

$$|\Theta_{\vec{k}} M_b \Psi_{\vec{j}} f(x)| \lesssim 2^{-\gamma|j_2 - k_2|} \mathcal{M}_S f(x).$$

Then taking the geometric mean of these two estimates, we have

$$|\Theta_{\vec{k}} M_b \Psi_{\vec{j}} f(x)| \lesssim 2^{-\gamma|j_1 - k_1|/2} 2^{-\gamma|j_2 - k_2|/2} \mathcal{M}_S f(x).$$

This completes the proof.  $\square$

Given a para-accretive function  $b$ , let  $S_k^b$  and  $D_k^b = S_{k+1}^b - S_k^b$  be the operators from (3.7). Theorem 2.3 in [Han94] says that there exist operators  $\tilde{D}_k^b$  for  $k \in \mathbb{Z}$  such that

$$(3.10) \quad \sum_{k \in \mathbb{Z}} \tilde{D}_k^b M_b D_k^b M_b f = f$$

in  $L^p(\mathbb{R}^n)$  for any function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $|f(x)| \lesssim \Phi_0^N(x)$  for some  $N > n$ ,  $|f(x) - f(y)| \lesssim |x - y|^\gamma$  for some  $\gamma > 0$ , and  $b f$  has mean zero. Furthermore,  $\tilde{D}_k^b$  is given by integration against its kernel  $\tilde{d}_k^b : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ ,

$$\tilde{D}_k^b f(x) = \int_{\mathbb{R}^n} \tilde{d}_k^b(x, y) f(y) dy,$$

and  $\tilde{d}_k^b$  satisfies

$$\begin{aligned} |\tilde{d}_k^b(x, y)| & \lesssim \Phi_k^{N+\gamma}(x - y), \\ |\tilde{d}_k^b(x, y) - \tilde{d}_k^b(x', y)| & \lesssim (2^k |x - x'|)^\gamma \left( \Phi_k^{N+\gamma}(x - y) + \Phi_k^{N+\gamma}(x' - y) \right), \\ \int_{\mathbb{R}^n} \tilde{d}_k^b(x, y) b(y) dy & = \int_{\mathbb{R}^n} \tilde{d}_k^b(x, y) b(x) dx = 0 \end{aligned}$$

for some  $N > n$  and  $0 < \gamma \leq 1$ .

**Lemma 3.2.** *Let  $b_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2 \in L^\infty(\mathbb{R}^{n_2})$  be para-accretive functions and  $D_{k_1}^{b_1}$  and  $D_{k_2}^{b_2}$  be the operators defined above. Also define  $D_{\vec{k}} = D_{k_1}^{b_1} D_{k_2}^{b_2}$  for  $\vec{k} \in \mathbb{Z}^2$ . Then*

$$\left\| \left( \sum_{\vec{k} \in \mathbb{Z}^2} |D_{\vec{k}} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for  $1 < p < \infty$  and  $f \in L^p(\mathbb{R}^n)$ .

This proof is essentially the same as the one due to R. Fefferman and Stein in Theorem 2 of [FS82]. We reproduce the argument to demonstrate that there are no problems that arise by introducing para-accretive perturbations.

*Proof.* We start by viewing the operator  $\{D_{k_1}^{b_1}\}$  defined initially from  $L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$  into  $L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))$  in the following way: for  $\{F_{k_2}\} \in L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$ , define

$$\{D_{k_1}^{b_1}\}(\{F_{k_2}\})(x_1) = \{D_{k_1}^{b_1} F_{k_2}(x_1)\}_{k_1, k_2 \in \mathbb{Z}}; \quad \text{for } x_1 \in \mathbb{R}^{n_1}.$$

Let  $\{F_{k_2}\} \in L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$ . For each  $k_2 \in \mathbb{Z}$ , we use the square function bound for  $D_{k_1}^{b_1}$  from [DJS85], and it follows that

$$\int_{\mathbb{R}^{n_1}} \sum_{k_1 \in \mathbb{Z}} |D_{k_1}^{b_1} F_{k_2}(x_1)|^2 dx_1 \lesssim \int_{\mathbb{R}^{n_1}} |F_{k_2}(x_1)|^2 dx_1.$$

Then it follows that

$$\begin{aligned} \|\{D_{k_1}^{b_1}\}(\{F_{k_2}\})\|_{L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))}^2 &= \sum_{k_2 \in \mathbb{Z}} \left( \int_{\mathbb{R}^{n_1}} \sum_{k_1 \in \mathbb{Z}} |D_{k_1}^{b_1} F_{k_2}(x_1)|^2 dx_1 \right) \\ &\lesssim \sum_{k_2 \in \mathbb{Z}} \left( \int_{\mathbb{R}^{n_1}} |F_{k_2}(x_1)|^2 dx_1 \right) = \|\{F_{k_2}\}\|_{L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}))}^2. \end{aligned}$$

That is,  $\{D_{k_1}^{b_1}\}$  is bounded from  $L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$  into  $L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))$ . Now the kernel of  $\{D_{k_1}^{b_1}\}$  is given by  $\{d_{k_1}^{b_1}(x_1, y_1)\} \in \mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}^2))$  for all  $x_1, y_1 \in \mathbb{R}^{n_1}$ , where  $\mathcal{L}(X, Y)$  for Banach spaces  $X$  and  $Y$  denotes the collection of all linear operators from  $X$  into  $Y$ . For fixed  $x_1, y_1 \in \mathbb{R}^{n_1}$ , the kernel  $\{d_{k_1}^{b_1}(x_1, y_1)\}$  is realized as a linear operator by the scalar multiplication:  $\{a_{k_2}\} \mapsto \{d_{k_1}^{b_1}(x_1, y_1) a_{k_2}\}_{(k_1, k_2) \in \mathbb{Z}^2}$ . Furthermore for  $x_1 \neq y_1$

$$\begin{aligned} \|\{d_{k_1}^{b_1}(x_1, y_1)\}\|_{\mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}^2))} &= \sup_{\|\{a_{k_2}\}\|_{\ell^2(\mathbb{Z})}=1} \|\{d_{k_1}^{b_1}(x_1, y_1) a_{k_2}\}\|_{\ell^2(\mathbb{Z}^2)} \\ &= \sup_{\|\{a_{k_2}\}\|_{\ell^2(\mathbb{Z})}=1} \|\{d_{k_1}^{b_1}(x_1, y_1)\}\|_{\ell^2(\mathbb{Z})} \|\{a_{k_2}\}\|_{\ell^2(\mathbb{Z})} \\ &= \|\{d_{k_1}^{b_1}(x_1, y_1)\}\|_{\ell^2(\mathbb{Z})} \lesssim \frac{1}{|x_1 - y_1|^{n_1}}. \end{aligned}$$

The last inequality is a well-known vector-valued Calderón-Zygmund kernel result, see e.g. Coifman-Meyer [CM78]. It also follows that

$$\begin{aligned} \|\{d_{k_1}^{b_1}(x_1, y_1)\} - \{d_{k_1}^{b_1}(x'_1, y_1)\}\|_{\mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}^2))} &\lesssim \frac{|x_1 - x'_1|^\gamma}{|x_1 - y_1|^{n_1 + \gamma}}; \text{ for } |x_1 - x'_1| < |x_1 - y_1|/2, \\ \|\{d_{k_1}^{b_1}(x_1, y_1)\} - \{d_{k_1}^{b_1}(x_1, y'_1)\}\|_{\mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}^2))} &\lesssim \frac{|y_1 - y'_1|^\gamma}{|x_1 - y_1|^{n_1 + \gamma}}; \text{ for } |y_1 - y'_1| < |x_1 - y_1|/2. \end{aligned}$$

Then  $\{D_{k_1}^{b_1}\}$  is bounded from  $L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$  into  $L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))$  for  $1 < p < \infty$  by the vector-valued Calderón-Zygmund theory developed by Benedek-Calderón-Panzone in [BCP62] and by Rubio de Francia-Ruiz-Torrea in [RdFRT83]. Alternatively, see Theorem 4.6.1 in Grafakos [Gra04] for a statement of the result applied here. Now we fix  $f \in L^p(\mathbb{R}^n)$  and define for  $x_2 \in \mathbb{R}^{n_2}$  and  $k_2 \in \mathbb{Z}$ ,

$$F_{k_2}^{x_2}(x_1) = D_{k_2}^{b_2} f(x) = \int_{\mathbb{R}^{n_2}} d_{k_2}^{b_2}(x_2, y_2) f(x_1, y_2) dy_2.$$

For almost every  $x_2 \in \mathbb{R}^{n_2}$ , we have  $\{F_{k_2}^{x_2}\} \in L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$  and hence

$$\begin{aligned} \int_{\mathbb{R}^{n_1}} \left( \sum_{\vec{k} \in \mathbb{Z}^2} |D_{\vec{k}} f(x)|^2 \right)^{\frac{p}{2}} dx_1 &= \int_{\mathbb{R}^{n_1}} \left( \sum_{\vec{k} \in \mathbb{Z}^2} |D_{k_1}^{b_1} F_{k_2}^{x_2}(x_1)|^2 \right)^{\frac{p}{2}} dx_1 \\ &= \|\{D_{k_1}^{b_1}\}(\{F_{k_2}^{x_2}\})\|_{L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))}^p \\ &\lesssim \|\{F_{k_2}^{x_2}\}\|_{L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))}^p = \int_{\mathbb{R}^{n_1}} \left( \sum_{k_2 \in \mathbb{Z}} |D_{k_2}^{b_2} f(x)|^2 \right)^{\frac{p}{2}} dx_1. \end{aligned} \tag{3.11}$$

Now integrate both sides of (3.11) in  $x_2$ , and using the square function bound for  $D_{k_2}^{b_2}$ , it follows that

$$\begin{aligned} \int_{\mathbb{R}^n} \left( \sum_{\vec{k} \in \mathbb{Z}^2} |D_{\vec{k}} f(x)|^2 \right)^{\frac{p}{2}} dx &\lesssim \int_{\mathbb{R}^{n_1}} \left[ \int_{\mathbb{R}^{n_2}} \left( \sum_{k_2 \in \mathbb{Z}} |D_{k_2}^{b_2} f(x)|^2 \right)^{\frac{p}{2}} dx_2 \right] dx_1 \\ &\lesssim \int_{\mathbb{R}^{n_1}} \left[ \int_{\mathbb{R}^{n_2}} |f(x)|^p dx_2 \right] dx_1 = \|f\|_{L^p(\mathbb{R}^n)}^p. \end{aligned}$$

This completes the proof.  $\square$

We now prove Theorem 5, but first we specify precisely which assumptions on  $\theta_{\vec{k}}$  are needed. One need not assume that  $\Theta_{\vec{k}}$  for  $\vec{k} \in \mathbb{Z}^2$  is a collection of biparameter Littlewood-Paley-Stein operators as initially stated in Theorem 5. Instead, we only need to assume that  $\theta_{\vec{k}}$  satisfies (3.1), (3.4), and (3.5). In short, we can remove the assumption that  $\theta_{\vec{k}}$  satisfies conditions (3.2) and (3.3) from Theorem 5. In particular, this means that the square function associated to  $\tilde{D}_{\vec{k}}^*$  is bounded as well: let  $\tilde{D}_{k_1}^{b_1}$  and  $\tilde{D}_{k_2}^{b_2}$  be the operators constructed in Theorem 2.3 from [Han94]. Define  $\tilde{D}_{\vec{k}} = \tilde{D}_{k_1}^{b_1} \tilde{D}_{k_2}^{b_2}$  for  $\vec{k} \in \mathbb{Z}^2$ , and it follows that

$$\left\| \left( \sum_{\vec{k} \in \mathbb{Z}^2} |\tilde{D}_{\vec{k}}^* f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p}$$

for all  $f \in L^p(\mathbb{R}^n)$  when  $1 < p < \infty$ . Before we prove Theorem 5, we prove a lemma analogous to the result in Theorem 2.3 from [Han94].

**Lemma 3.3.** *Let  $b_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2 \in L^\infty(\mathbb{R}^{n_2})$  be para-accretive functions and  $b(x) = b_1(x_1)b_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^n$ . For  $j = 1, 2$  let  $D_{k_j}^{b_j}$  be as in (3.7) and  $\tilde{D}_{k_j}^{b_j}$  be as in (3.10) from Theorem 2.3 in [Han94]. Define  $E_{k_j}^{b_j} = \tilde{D}_{k_j} M_{b_j} D_{k_j}^{b_j}$  for  $k_j \in \mathbb{Z}$  and  $j = 1, 2$ . For any differentiable compactly supported function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  such that*

$$\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = 0$$

for  $x = (x_1, x_2) \in \mathbb{R}^n$ , we have the following convergence

$$\lim_{T \rightarrow \infty} \left\| \sum_{|j_1| < T, |j_2| < N_T} E_{j_1} M_{b_1} f - f \right\|_{L^p(\mathbb{R}^n)} = 0$$

for some sequence  $N_T \geq T$ .

*Proof.* Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  be differentiable and compactly supported such that

$$\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = 0.$$

For each  $x_2 \in \mathbb{R}^{n_2}$ ,  $f(\cdot, x_2)$  is differentiable, compactly supported, and  $b_1 \cdot f(\cdot, x_2)$  has mean zero. Then by Theorem 2.3 in [Han94], for every  $x_2 \in \mathbb{R}^{n_2}$

$$\lim_{T \rightarrow \infty} \left\| \sum_{|j_1| < T} E_{j_1} M_{b_1} f(\cdot, x_2) - f(\cdot, x_2) \right\|_{L^p(\mathbb{R}^{n_1})} = 0$$

Since  $f$  is compactly supported and the above quantity is bounded uniformly in  $T$ , it follows by dominated convergence that

$$(3.12) \quad \lim_{T \rightarrow \infty} \left\| \sum_{|j_1| < T} E_{j_1} M_{b_1} f - f \right\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^{n_2}} \lim_{T \rightarrow \infty} \left\| \sum_{|j_1| < T} E_{j_1} M_{b_1} f(\cdot, x_2) - f(\cdot, x_2) \right\|_{L^p(\mathbb{R}^{n_1})}^p dx_2 = 0.$$

Define for each  $T > 0$

$$F_T^{x_1}(x_2) = \sum_{|j_1| < T} E_{j_1} M_{b_1} f(x_1, x_2).$$

It follows that

$$|F_T^{x_1}(x_2)| \leq \sum_{|j_1| < T} |E_{j_1} M_{b_1} f(x_1, x_2)| \leq 2T \mathcal{M}_1 f(x) \leq 2T \sup_{x_1 \in \mathbb{R}^{n_1}} |f(x_1, x_2)|.$$

Therefore  $F_T^{x_1} : \mathbb{R}^{n_2} \rightarrow \mathbb{C}$  is bounded (depending on  $T$ ) and compactly supported. Furthermore

$$\begin{aligned} |F_T^{x_1}(x_2) - F_T^{x_1}(y_2)| &\leq \sum_{|j_1| < T} |E_{j_1} M_{b_1} f(x_1, x_2) - f(x_1, y_2)| \\ &\leq \sum_{|j_1| < T} \int_{\mathbb{R}^{n_2}} |\tilde{d}_{j_1}^{b_1}(x_2, u_2) - \tilde{d}_{j_1}^{b_1}(y_2, u_2)| |M_{b_1} D_{j_1}^{b_1} M_{b_1} f(x_1, u_2)| du_2 \\ &\lesssim \sum_{|j_1| < T} \int_{\mathbb{R}^{n_2}} (2^{j_1} |x_2 - y_2|)^{\gamma} |D_{j_1}^{b_1} M_{b_1} f(x_1, u_2)| du_2 \\ &\lesssim 2^T |x_2 - y_2|^{\gamma} \sum_{|j_1| < T} \|D_{j_1}^{b_1} M_{b_1} f(x_1, \cdot)\|_{L^1(\mathbb{R}^{n_2})} \\ &\leq 2^T |x_2 - y_2|^{\gamma} \sum_{|j_1| < T} \|f(x_1, \cdot)\|_{L^1(\mathbb{R}^{n_2})} \leq T 2^{T+1} \|f(x_1, \cdot)\|_{L^1(\mathbb{R}^{n_2})} |x_2 - y_2|^{\gamma}. \end{aligned}$$

Finally, we have that

$$\int_{\mathbb{R}^{n_2}} F_T^{x_1} b_2(x_2) dx_2 = \sum_{|j_1| < T} E_{j_1} M_{b_1} \int_{\mathbb{R}^{n_2}} f(x_1, x_2) b_2(x_2) dx_2 = 0.$$

Then by Theorem 2.3 from [Han94], it follow that

$$\lim_{N \rightarrow \infty} \left\| \sum_{|j_2| < N} E_{j_2} M_{b_2} F_T^{x_1} - F_T^{x_1} \right\|_{L^p(\mathbb{R}^{n_2})} = 0.$$

Then by dominated convergence

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\| \sum_{|j_1| < T, |j_2| < N} E_{j_2} M_{b_2} f - \sum_{|j_1| < T} E_{j_1} M_{b_1} f \right\|_{L^p(\mathbb{R}^{n_2})}^p \\ (3.13) \quad = \int_{\mathbb{R}^{n_1}} \lim_{N \rightarrow \infty} \left\| \sum_{|j_2| < N} E_{j_2} M_{b_2} F_T^{x_1} - F_T^{x_1} \right\|_{L^p(\mathbb{R}^{n_2})}^p dx_1 = 0. \end{aligned}$$

For each  $T > 0$ , using (3.13) there exists  $N_T > T$  such that

$$\left\| \sum_{|j_1| < T, |j_2| < N_T} E_{j_2} M_{b_2} f - \sum_{|j_1| < T} E_{j_1} M_{b_1} f \right\|_{L^p(\mathbb{R}^{n_2})} < \frac{1}{T}.$$

This defines the sequence  $N_T$ , and so now we verify the conclusion of Lemma 3.3. Let  $\varepsilon > 0$ . Fix  $M > \frac{2}{\varepsilon}$  large enough so that for  $T > M$

$$\left\| \sum_{|j_1| < T} E_{j_1} M_{b_1} f - f \right\|_{L^p(\mathbb{R}^n)} < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} \left\| \sum_{|j_1| < T, |j_2| < N_T} E_{\vec{j}} M_b f - f \right\|_{L^p(\mathbb{R}^n)} &= \left\| \sum_{|j_1| < T, |j_2| < N_T} E_{\vec{j}} M_b f - \sum_{|j_1| < T} E_{j_1} M_{b_1} f \right\|_{L^p(\mathbb{R}^n)} + \left\| \sum_{|j_1| < T} E_{j_1} M_{b_1} f - f \right\|_{L^p(\mathbb{R}^n)} \\ &< \frac{1}{T} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This completes the proof.  $\square$

Now we prove Theorem 5.

*Proof.* Let  $b(x) = b_1(x_1)b_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^n$ , and  $f, g_{\vec{k}}$  be differentiable, compactly supported such that

$$\int_{\mathbb{R}^{n_1}} f(x)b(x)dx_1 = \int_{\mathbb{R}^{n_2}} f(x)b(x)dx_2 = 0$$

and

$$\left\| \left( \sum_{\vec{k} \in \mathbb{Z}^2} |g_{\vec{k}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \leq 1.$$

Let  $R > 1$ , and define

$$\Lambda_R(f) = \sum_{|k_1|, |k_2| < R} \left| \int_{\mathbb{R}^n} \Theta_{\vec{k}} M_b f(x) g_{\vec{k}}(x) dx \right|,$$

which satisfies

$$(3.14) \quad 0 \leq \Lambda_R(f) \lesssim \int_{\mathbb{R}^n} \mathcal{M}_S f(x) \sum_{|k_1|, |k_2| < R} |g_{\vec{k}}(x)| dx \lesssim R \|f\|_{L^p}.$$

Let  $S_{k_j}^{b_j}, D_{k_j}^{b_j} = S_{k_j+1}^{b_j} - S_{k_j}^{b_j}, \tilde{D}_{k_j}^{b_j}$ , and  $D_{\vec{k}} = D_{k_1}^{b_1} D_{k_2}^{b_2}$  be the operators defined in (3.7). Also define  $E_{k_j}^{b_j} = \tilde{D}_{k_j}^{b_j} M_{b_j} D_{k_j}^{b_j}$  and  $E_{\vec{k}} = E_{k_1}^{b_1} E_{k_2}^{b_2}$ , where  $\tilde{D}_{k_j}^{b_j}$  are the operators from (3.10) that were constructed in Theorem 2.3 of [Han94]. Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  be continuous, compactly supported such that

$$\int_{\mathbb{R}^{n_1}} f(x)b_1(x_1)dx_1 = \int_{\mathbb{R}^{n_2}} f(x)b_2(x_2)dx_2 = 0$$

for all  $x = (x_1, x_2) \in \mathbb{R}^n$ . For  $T > 1$  it follows that

$$\begin{aligned} \Lambda_R(f) &\leq \sum_{|k_1|, |k_2| < R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b - \Theta_{\vec{k}} M_b \left( \sum_{|j_1| < T, |j_2| < N_T} E_{\vec{j}} M_b \right) \right] f(x) g_{\vec{k}}(x) dx \right| \\ &\quad + \sum_{|k_1|, |k_2| < R} \left| \sum_{|j_1| < T, |j_2| < N_T} \int_{\mathbb{R}^n} \Theta_{\vec{k}} M_b E_{\vec{j}} M_b f(x) g_{\vec{k}}(x) dx \right| = I_T + II_T. \end{aligned}$$

where  $N_T$  are chosen as in Lemma 3.3. We first estimate  $I_T$  using (3.14):

$$\begin{aligned} I_T &= \sum_{|k_1|, |k_2| < R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b \left( f(x) - \sum_{|j_1| < T, |j_2| < N_T} E_{\vec{j}} M_b f(x) \right) \right] g_{\vec{k}}(x) dx \right| \\ &\leq \Lambda_R \left( f - \sum_{|j_1| < T, |j_2| < N_T} E_{\vec{j}} M_b f \right) \lesssim R \left\| f - \sum_{|j_1| < T, |j_2| < N_T} E_{\vec{j}} M_b f \right\|_{L^p}, \end{aligned}$$

which tends to 0 as  $T \rightarrow \infty$  by Lemma 3.3. Now we estimate  $II_T$  by putting the absolute value inside and summing more terms,

$$II_T \leq \sum_{\vec{k}, \vec{j} \in \mathbb{Z}^2} \int_{\mathbb{R}^n} |\Theta_{\vec{k}} M_b E_{\vec{j}} M_b f(x) g_{\vec{k}}(x)| dx,$$



So we now estimate  $II_T$ . By Lemma 3.1, there exists  $\varepsilon > 0$  such that

$$|\Theta_{\vec{k}} M_b E_{\vec{j}} f(x)| \lesssim 2^{-\varepsilon|k_1-j_1|} 2^{-\varepsilon|k_2-j_2|} \mathcal{M}_S D_{\vec{j}} M_b f(x).$$

Then it follows that

$$\begin{aligned} \Lambda_R(f) &\leq \int_{\mathbb{R}^n} \sum_{\vec{j}, \vec{k} \in \mathbb{Z}^2} |\Theta_{\vec{k}} M_b E_{\vec{j}} M_b f(x) g_{\vec{k}}(x)| dx \\ &\lesssim \int_{\mathbb{R}^n} \sum_{\vec{j}, \vec{k} \in \mathbb{Z}^2} 2^{-\frac{\varepsilon}{2}(|k_1-j_1|+|k_2-j_2|)} \mathcal{M}_S \left( D_{\vec{j}} M_b f \right) (x) |g_{\vec{k}}(x)| dx \\ &\leq \left\| \left( \sum_{\vec{j}, \vec{k} \in \mathbb{Z}^2} 2^{-\frac{\varepsilon}{2}(|k_1-j_1|+|k_2-j_2|)} \left[ \mathcal{M}_S \left( D_{\vec{j}} M_b f \right) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \left\| \left( \sum_{\vec{j}, \vec{k} \in \mathbb{Z}^2} 2^{-\frac{\varepsilon}{2}(|k_1-j_1|+|k_2-j_2|)} |g_{\vec{k}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \\ &\lesssim \left\| \left( \sum_{\vec{j} \in \mathbb{Z}^2} \left[ \mathcal{M}_S \left( D_{\vec{j}} M_b f \right) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \left\| \left( \sum_{\vec{k} \in \mathbb{Z}^2} |g_{\vec{k}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim \left\| \left( \sum_{\vec{j} \in \mathbb{Z}^2} |D_{\vec{j}} M_b f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

In the last two lines we use the Fefferman-Stein strong maximal function bound from [FS82] twice and the multiparameter Littlewood-Paley bound from Lemma 3.2. The estimate for general functions  $f \in L^p(\mathbb{R}^n)$  follows by density.  $\square$

Next we prove a sort of dual pairing bound for biparameter Littlewood-Paley-Stein operators. This is the estimate that we use to bound the truncations of singular integral operators in the next section.

**Proposition 3.1.** *Let  $\Theta_{\vec{k}}$  be a collection of biparameter Littlewood-Paley-Stein operators with kernels  $\theta_{\vec{k}}$  for  $\vec{k} \in \mathbb{Z}^2$  and  $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$  be para-accretive functions. If*

$$\int_{\mathbb{R}^{n_j}} \theta_{\vec{k}}(x, y) b_j(y) dy_j = \int_{\mathbb{R}^{n_j}} \theta_{\vec{k}}(x, y) \tilde{b}_j(x_j) dx_j = 0$$

for  $j = 1, 2$ , then for all  $f \in L^p(\mathbb{R}^n)$  and  $g \in L^{p'}(\mathbb{R}^n)$

$$\sum_{k_1, k_2 \in \mathbb{Z}} \left| \int_{\mathbb{R}^2} \Theta_{\vec{k}} M_b f(x) \tilde{b}(x) g(x) dx \right| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)},$$

where  $b(x) = b_1(x_1)b_2(x_2)$  and  $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^n$ .

*Proof.* Let  $f, g$  be differentiable, compactly supported functions such that

$$\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = \int_{\mathbb{R}^{n_1}} g(x) \tilde{b}(x) dx_1 = \int_{\mathbb{R}^{n_2}} g(x) \tilde{b}(x) dx_2 = 0.$$

Define for  $R > 1$

$$\Lambda_R(f, g) = \sum_{|k_1|, |k_2| < R} \left| \int_{\mathbb{R}^n} \Theta_{\vec{k}} M_b f(x) \tilde{b}(x) g(x) dx \right|,$$

which satisfies

$$(3.15) \quad 0 \leq \Lambda_R(f, g) \lesssim \sum_{|k_1|, |k_2| < R} \|\mathcal{M}_S f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)} \lesssim R^2 \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Let  $S_{k_j}^{b_j}, D_{k_j}^{b_j} = S_{k_j+1}^{b_j} - S_{k_j}^{b_j}$ ,  $\tilde{D}_{k_j}^{b_j}, \tilde{D}_{k_j}^b = D_{k_1}^{b_1} D_{k_2}^{b_2}$ , and  $\tilde{D}_{k_j}^b = \tilde{D}_{k_1}^{b_1} \tilde{D}_{k_2}^{b_2}$  be the operators defined in (3.7). Also define  $E_{k_j}^{b_j} = \tilde{D}_{k_j}^{b_j} M_{b_j} D_{k_j}^{b_j}$  and  $E_{k_j}^b = E_{k_1}^{b_1} E_{k_2}^{b_2}$ , where  $\tilde{D}_{k_j}^{b_j}$  are the operators constructed in Theorem 2.3 in [Han94]. We also construct the corresponding operators with  $b_j$  replaced by  $\tilde{b}_j$ . Then for  $f, g \in C_0^\delta(\mathbb{R}^n)$  for some  $0 < \delta \leq 1$  where  $b f$

and  $\tilde{b}g$  have mean zero in both  $x_1$  and  $x_2$ , it follows that

$$\Lambda_R(f, g) \leq \limsup_{T \rightarrow \infty} I_T + II_T + III_T,$$

where

$$\begin{aligned} I_T &= \sum_{|k_1|, |k_2| < R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b - \Theta_{\vec{k}} M_b \left( \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b \right) \right] f(x) M_{\tilde{b}} g(x) dx \right|, \\ II_T &= \sum_{|k_1|, |k_2| < R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b \left( \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b \right) \right. \right. \\ &\quad \left. \left. - \left( \sum_{|m_1| < T, |m_2| < M_T} E_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} \right) \Theta_{\vec{k}} M_b \left( \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b \right) \right] f(x) M_{\tilde{b}} g(x) dx \right|, \\ III_T &= \sum_{|k_1|, |k_2| < R} \left| \sum_{|j_1| < T, |j_2| < N_T, |m_1| < T, |m_2| < M_T} \int_{\mathbb{R}^n} E_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} \Theta_{\vec{k}} M_b E_j^b M_b f(x) M_{\tilde{b}} g(x) dx \right|, \end{aligned}$$

where  $N_T$  and  $M_T$  are chosen as in Lemma 3.3 for  $f$  and  $g$  respectively. We first estimate  $I_T$  using (3.15) and Lemma 3.3:

$$\begin{aligned} I_T &= \sum_{|k_1|, |k_2| < R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b \left( f(x) - \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b f(x) \right) \right] M_{\tilde{b}} g(x) dx \right| \\ &\leq \Lambda_R \left( f - \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b f, g \right) \lesssim R \left\| f - \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b f \right\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}, \end{aligned}$$

which tends to 0 as  $T \rightarrow \infty$ . Now we estimate  $II_T$  again using (3.15) and Lemma 3.3,

$$\begin{aligned} II_T &= \sum_{|k_1|, |k_2| < R} \left| \int_{\mathbb{R}^n} \left[ \mathbf{I} - \sum_{|m_1| < T, |m_2| < M_T} E_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} \right] \Theta_{\vec{k}} M_b \left( \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b \right) f(x) M_{\tilde{b}} g(x) dx \right| \\ &= \Lambda_R \left( \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b f, g - \sum_{|m_1| < T, |m_2| < M_T} E_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} g \right) \\ &\lesssim R \left\| \sum_{|j_1| < T, |j_2| < N_T} E_j^b M_b f \right\|_{L^p(\mathbb{R}^n)} \left\| g - \sum_{|m_1| < T, |m_2| < M_T} E_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} g \right\|_{L^{p'}(\mathbb{R}^n)} \\ &\lesssim R \|f\|_{L^p(\mathbb{R}^n)} \left\| g - \sum_{|m_1| < T, |m_2| < M_T} E_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} g \right\|_{L^{p'}(\mathbb{R}^n)}, \end{aligned}$$

where  $\mathbf{I}$  is the identity operator. This term also tends to 0 as  $T \rightarrow \infty$  by Lemma 3.3. So we are left with the third term, to estimate  $\Lambda_R$

$$\begin{aligned} \Lambda_R(f, g) &\leq \limsup_{T \rightarrow \infty} \sum_{|k_1|, |k_2| < R} \left| \sum_{|j_1| < T, |j_2| < N_T, |m_1| < T, |m_2| < M_T} \int_{\mathbb{R}^n} E_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} \Theta_{\vec{k}} M_b E_j^b M_b f(x) M_{\tilde{b}} g(x) dx \right| \\ (3.16) \quad &\leq \sum_{\vec{k}, \vec{j}, \vec{m} \in \mathbb{Z}^2} \left| \int_{\mathbb{R}^n} M_b D_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} \Theta_{\vec{k}} M_b E_j^b M_b f(x) (\tilde{D}_{\vec{m}}^{\tilde{b}})^* M_{\tilde{b}} g(x) dx \right|. \end{aligned}$$

So we now estimate (3.16). By Lemma 3.1, there exists  $\varepsilon > 0$  such that

$$\begin{aligned} |D_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} \Theta_{\vec{k}} M_b E_j^b f(x)| &\lesssim 2^{-\varepsilon|m_1-k_1|} 2^{-\varepsilon|m_2-k_2|} \mathcal{M}_S^2 D_j^b f(x), \quad \text{and} \\ |D_{\vec{m}}^{\tilde{b}} M_{\tilde{b}} \Theta_{\vec{k}} M_b E_j^b f(x)| &\lesssim \mathcal{M}_S(\Theta_{\vec{k}} M_b E_j^b f)(x) \lesssim 2^{-\varepsilon|k_1-j_1|} 2^{-\varepsilon|k_2-j_2|} \mathcal{M}_S^2 D_j^b f(x). \end{aligned}$$

Therefore we also have

$$(3.17) \quad |D_{\tilde{m}}^b M_{\tilde{b}} \Theta_{\tilde{k}} M_b E_{\tilde{j}}^b f(x)| \lesssim 2^{-\frac{\varepsilon}{2}(|m_1-k_1|+|m_2-k_2|+|k_1-j_1|+|k_2-j_2|)} \mathcal{M}_S^2 D_{\tilde{j}}^b f(x).$$

Using (3.17) we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \sum_{\tilde{j}, \tilde{k}, \tilde{m} \in \mathbb{Z}^2} |M_{\tilde{b}} D_{\tilde{m}}^b M_{\tilde{b}} \Theta_{\tilde{k}} M_b E_{\tilde{j}}^b f(x) (\tilde{D}_{\tilde{m}}^b)^* M_{\tilde{b}} g(x)| dx \lesssim \int_{\mathbb{R}^n} \sum_{\tilde{j}, \tilde{k}, \tilde{m} \in \mathbb{Z}^2} 2^{-\frac{\varepsilon}{2}(|m_1-k_1|+|m_2-k_2|+|k_1-j_1|+|k_2-j_2|)} \\ & \quad \times \mathcal{M}_S^2 \left( D_{\tilde{j}}^b M_b f \right) (x) (\tilde{D}_{\tilde{m}}^b)^* M_{\tilde{b}} g(x) | dx \\ & \leq \left\| \left( \sum_{\tilde{j}, \tilde{k}, \tilde{m} \in \mathbb{Z}^2} 2^{-\frac{\varepsilon}{2}(|m_1-k_1|+|m_2-k_2|+|k_1-j_1|+|k_2-j_2|)} \left[ \mathcal{M}_S^2 \left( D_{\tilde{j}}^b M_b f \right) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \\ & \quad \times \left\| \left( \sum_{\tilde{j}, \tilde{k}, \tilde{m} \in \mathbb{Z}^2} 2^{-\frac{\varepsilon}{2}(|m_1-k_1|+|m_2-k_2|+|k_1-j_1|+|k_2-j_2|)} |(\tilde{D}_{\tilde{m}}^b)^* M_{\tilde{b}} g|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \\ & \lesssim \left\| \left( \sum_{\tilde{j} \in \mathbb{Z}^2} \left[ \mathcal{M}_S^2 \left( D_{\tilde{j}}^b M_b f \right) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \left\| \left( \sum_{\tilde{m} \in \mathbb{Z}^2} |(\tilde{D}_{\tilde{m}}^b)^* M_{\tilde{b}} g|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \\ & \lesssim \left\| \left( \sum_{\tilde{j} \in \mathbb{Z}^2} |D_{\tilde{j}}^b M_b f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}. \end{aligned}$$

In the last two lines we use the Fefferman-Stein maximal function bound from [FS82] twice and the biparameter Littlewood-Paley-Stein bound proved in Theorem 5. Recall that the square function associated to  $(\tilde{D}_{\tilde{m}}^b)^*$  is bounded on  $L^p(\mathbb{R}^n)$  for  $1 < p < \infty$ . The estimate for general functions  $f \in L^p(\mathbb{R}^n)$  and  $g \in L^{p'}(\mathbb{R}^n)$  follows by density.  $\square$

#### 4. A BIPARAMETER TB THEOREM

We define the class of test functions that will be used to define biparameter singular integral operators. Define  $C_0^{0,\delta}(\mathbb{R}^n)$  to be the collection of all  $\delta$ -Hölder continuous, compactly supported functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  with norm

$$\|f\|_{\delta} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\delta}} < \infty.$$

Since  $C_0^{0,\delta}(\mathbb{R}^n)$  is made up of compactly supported functions, it follows that  $\|\cdot\|_{\delta}$  is a norm, and we endow  $C_0^{0,\delta}(\mathbb{R}^n)$  the topology generated by the norm  $\|\cdot\|_{\delta}$ . Given a function  $b \in L^{\infty}(\mathbb{R}^n)$  such that  $b^{-1} \in L^{\infty}(\mathbb{R}^n)$ , let  $bC_0^{0,\delta}(\mathbb{R}^n)$  be the collection of functions  $bf$  such that  $f \in C_0^{0,\delta}(\mathbb{R}^n)$ . We define  $\|bf\|_{b,\delta} = \|f\|_{\delta}$  for  $bf \in bC_0^{0,\delta}(\mathbb{R}^n)$ , and endow  $bC_0^{0,\delta}(\mathbb{R}^n)$  the topology generated by the norm  $\|\cdot\|_{b,\delta}$ . Finally, given a function space  $X$ , we define  $X'$  to be the continuous dual of  $X$  with the weak\* topology. In our situation, we will primarily use this definition for  $X = bC_0^{0,\delta}(\mathbb{R}^n)$ .

**Definition 4.1.** We say that  $K$  a standard biparameter kernel on  $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  if

$$(4.1) \quad |K(x, y)| \lesssim \frac{1}{|x_1 - y_1|^{n_1} |x_2 - y_2|^{n_2}} \quad \text{for } |x_1 - y_1|, |x_2 - y_2| \neq 0$$

$$(4.2) \quad |K(x, y) - K(x'_1, x_2, y) - K(x_1, x'_2, y) + K(x'_1, x'_2, y)| \lesssim \frac{|x_1 - x'_1|^{\gamma} |x_2 - x'_2|^{\gamma}}{|x_1 - y_1|^{n_1+\gamma} |x_2 - y_2|^{n_2+\gamma}}$$

whenever  $|x_1 - x'_1| < |x_1 - y_1|/2$  and  $|x_2 - x'_2| < |x_2 - y_2|/2$ ,

$$(4.3) \quad |K(x, y) - K(x, y'_1, y_2) - K(x, y_1, y'_2) + K(x, y'_1, y'_2)| \lesssim \frac{|y_1 - y'_1|^\gamma |y_2 - y'_2|^\gamma}{|x_1 - y_1|^{n_1 + \gamma} |x_2 - y_2|^{n_2 + \gamma}}$$

whenever  $|y_1 - y'_1| < |x_1 - y_1|/2$  and  $|y_2 - y'_2| < |x_2 - y_2|/2$ ,

$$(4.4) \quad |K(x, y) - K(x, y'_1, y_2) - K(x_1, x'_2, y) + K(x_1, x'_2, y'_1, y_2)| \lesssim \frac{|y_1 - y'_1|^\gamma |x_2 - x'_2|^\gamma}{|x_1 - y_1|^{n_1 + \gamma} |x_2 - y_2|^{n_2 + \gamma}}$$

whenever  $|y_1 - y'_1| < |x_1 - y_1|/2$  and  $|x_2 - x'_2| < |x_2 - y_2|/2$ ,

$$(4.5) \quad |K(x, y) - K(x, y_1, y'_2) - K(x'_1, x_2, y) + K(x'_1, x_2, y_1, y'_2)| \lesssim \frac{|x_1 - x'_1|^\gamma |y_2 - y'_2|^\gamma}{|x_1 - y_1|^{n_1 + \gamma} |x_2 - y_2|^{n_2 + \gamma}}$$

whenever  $|x_1 - x'_1| < |x_1 - y_1|/2$  and  $|y_2 - y'_2| < |x_2 - y_2|/2$ .

Let  $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$  be para-accretive functions and define  $b(x) = b_1(x_1)b_2(x_2)$  and  $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^n$ . A linear operator  $T$  that is continuous from  $bC_0^{0, \delta}(\mathbb{R}^n)$  into  $(\tilde{b}C_0^{0, \delta}(\mathbb{R}^n))'$  for some  $0 < \delta \leq 1$  is a biparameter singular integral operator of Calderón-Zygmund type associated to  $b, \tilde{b}$  if

$$\langle M_{\tilde{b}} T M_b f, g \rangle = \int_{\mathbb{R}^{2n}} K(x, y) f(y) g(x) \tilde{b}(x) b(y) dx dy$$

is an absolutely convergent integral whenever  $f, g \in C_0^{0, \delta}(\mathbb{R}^n)$  and

$$\bigcup_{x_1, y_1 \in \mathbb{R}^{n_1}} \text{supp}(f(y_1, \cdot)) \cap \text{supp}(g(x_1, \cdot)) = \bigcup_{x_2, y_2 \in \mathbb{R}^{n_2}} \text{supp}(f(\cdot, y_2)) \cap \text{supp}(g(\cdot, x_2)) = \emptyset.$$

**Definition 4.2.** A function  $\phi \in C_0^\infty(\mathbb{R}^n)$  is a normalized bump of order  $m \in \mathbb{N}$  if  $\text{supp}(\phi) \subset B(0, 1) \subset \mathbb{R}^n$  and for all  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| \leq m$

$$\|\partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \leq 1.$$

Let  $T$  be a biparameter singular integral operator of Calderón-Zygmund type associated to  $b(x) = b_1(x_1)b_2(x_2)$  and  $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^n$ , where  $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$  are para-accretive functions. We say  $T$  satisfies the biparameter weak boundedness property if there exists  $m \in \mathbb{N}$  such that the following holds: let  $\phi_j, \psi_j \in C_0^\infty(\mathbb{R}^{n_j})$  be normalized bumps of order  $m$ . Let  $x = (x_1, x_2) \in \mathbb{R}^n$  and  $R_1, R_2 > 0$ . Assume that either  $b_1 \phi_1^{x_1, R_1}$  or  $\tilde{b}_1 \psi_1^{x_1, R_1}$  has mean zero and that either  $b_2 \phi_2^{x_2, R_2}$  or  $\tilde{b}_2 \psi_2^{x_2, R_2}$  has mean zero. Then

$$(4.6) \quad \left| \left\langle M_{\tilde{b}} T M_b (\phi_1^{x_1, R_1} \otimes \phi_2^{x_2, R_2}), \psi_1^{x_1, R_1} \otimes \psi_2^{x_2, R_2} \right\rangle \right| \lesssim R_1^{n_1} R_2^{n_2},$$

where  $\phi_j^{x_j, R_j}(u_j) = \phi\left(\frac{u_j - x_j}{R_j}\right)$ .

**Definition 4.3.** Let  $T$  be a biparameter singular integral operator of Calderón-Zygmund type associated to  $b(x) = b_1(x_1)b_2(x_2)$  and  $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^n$ , where  $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$  are para-accretive functions. We say  $T$  satisfies the mixed biparameter weak boundedness property if there exists  $m \in \mathbb{N}$  and  $0 < \gamma \leq 1$  such that the following two conditions hold: (1) Let be  $R_1, R_2 > 0$ ,  $x_1, y_1 \in \mathbb{R}^{n_1}$  with  $|x_1 - y_1| > 4R_1$ , and  $x_2 \in \mathbb{R}^{n_2}$  and let  $\phi_j, \psi_j \in C_0^\infty(\mathbb{R}^{n_j})$  be normalized bumps of order  $m$ . Then

$$(4.7) \quad \left| \left\langle M_{\tilde{b}} T M_b (\phi_1^{y_1, R_1} \otimes \phi_2^{x_2, R_2}), \psi_1^{x_1, R_1} \otimes \psi_2^{x_2, R_2} \right\rangle \right| \lesssim \frac{R_1^{n_1} R_2^{n_2}}{(R_1^{-1} |x_1 - y_1|)^{n_1}}.$$

Further assume that either  $b_1 \phi_1^{y_1, R_1}$  or  $\tilde{b}_1 \psi_1^{x_1, R_1}$  has mean zero and that either  $b_2 \phi_2^{x_2, R_2}$  or  $\tilde{b}_2 \psi_2^{x_2, R_2}$  has mean zero. Then

$$(4.8) \quad \left| \left\langle M_{\tilde{b}} T M_b (\phi_1^{y_1, R_1} \otimes \phi_2^{x_2, R_2}), \psi_1^{x_1, R_1} \otimes \psi_2^{x_2, R_2} \right\rangle \right| \lesssim \frac{R_1^{n_1} R_2^{n_2}}{(R_1^{-1} |x_1 - y_1|)^{n_1 + \gamma}}.$$

(2) Let be  $R_1, R_2 > 0$ ,  $x_2, y_2 \in \mathbb{R}^{n_2}$  with  $|x_2 - y_2| > 4R_2$ , and  $x_1 \in \mathbb{R}^{n_1}$  and let  $\phi_j, \psi_j \in C_0^\infty(\mathbb{R}^{n_j})$  be normalized bumps of order  $m$ . Then

$$(4.9) \quad \left| \left\langle M_{\tilde{b}} T M_b (\phi_1^{x_1, R_1} \otimes \phi_2^{y_2, R_2}), \psi_1^{x_1, R_1} \otimes \psi_2^{y_2, R_2} \right\rangle \right| \lesssim \frac{R_1^{n_1} R_2^{n_2}}{(R_2^{-1} |x_2 - y_2|)^{n_2}}.$$

Further assume that either  $b_1\varphi_1^{x_1,R_1}$  or  $\tilde{b}_1\psi_1^{x_1,R_1}$  has mean zero and that either  $b_2\varphi_2^{y_2,R_2}$  or  $\tilde{b}_2\psi_2^{y_2,R_2}$  has mean zero. Then,

$$(4.10) \quad \left| \left\langle M_{\tilde{b}}TM_b(\varphi_1^{x_1,R_1} \otimes \varphi_2^{y_2,R_2}), \psi_1^{x_1,R_1} \otimes \psi_2^{y_2,R_2} \right\rangle \right| \lesssim \frac{R_1^{n_1}R_2^{n_2}}{(R_2^{-1}|x_2-y_2|)^{n_2+\gamma}}.$$

**Lemma 4.1.** *Suppose  $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$  and  $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$  are para-accretive functions, and define  $b(x) = b_1(x_1)b_2(x_2)$  and  $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^n$ . Let  $T$  be a biparameter singular integral operator of Calderón-Zygmund type associated to  $b$  and  $\tilde{b}$  with standard biparameter kernel  $K$ . Also assume that  $M_{\tilde{b}}TM_b$  satisfies the biparameter weak boundedness and the mixed weak boundedness properties. Define  $\Theta_{\vec{k}}$  for  $\vec{k} \in \mathbb{Z}^2$  by integration against its kernel  $\theta_{\vec{k}}$ , as in (3.6), where*

$$(4.11) \quad \theta_{\vec{k}}(x, y) = \left\langle M_{\tilde{b}}TM_b(s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)), d_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot) \right\rangle.$$

Then  $\Theta_{\vec{k}}$  for  $\vec{k} \in \mathbb{Z}^2$  is a collection of Littlewood-Paley-Stein operators and

$$\int_{\mathbb{R}^{n_1}} \theta_{\vec{k}}(x, y) \tilde{b}_1(x_1) dx_1 = \int_{\mathbb{R}^{n_2}} \theta_{\vec{k}}(x, y) \tilde{b}_2(x_2) dx_2 = 0.$$

*Proof.* Fix  $x, y \in \mathbb{R}^n$  such that  $|x_1 - y_1| \leq 2^{-k_1+2}$  and  $|x_2 - y_2| \leq 2^{-k_2+2}$ . Then using (4.6)

$$\begin{aligned} |\theta_{\vec{k}}(x, y)| &= 2^{2k_1n_1} 2^{2k_2n_2} \left| \left\langle M_{\tilde{b}}TM_b \left( \phi_1^{\frac{x_1+y_1}{2}, 2^{-k_1+2}} \otimes \phi_2^{\frac{x_2+y_2}{2}, 2^{-k_2+2}} \right), \phi_3^{\frac{x_1+y_1}{2}, 2^{-k_1+2}} \otimes \phi_4^{\frac{x_2+y_2}{2}, 2^{-k_2+2}} \right\rangle \right| \\ &\lesssim 2^{k_1n_1} 2^{k_2n_2} \lesssim \Phi_{k_1}^{n_1+\gamma}(x_1 - y_1) \Phi_{k_2}^{n_2+\gamma}(x_2 - y_2). \end{aligned}$$

where  $\phi_1, \phi_2, \phi_3, \phi_4$  are normalized bumps of order  $m$  (up to a constant multiple independent of  $x, y$ , and  $\vec{k}$ ) of the form

$$\begin{aligned} \phi_1(u_1) &= 2^{-k_1n_1} s_{k_1}^{b_1} \left( 2^{-k_1+2} u_1 + \frac{x_1+y_1}{2}, y_1 \right), & \phi_2(u_2) &= 2^{-k_2n_2} s_{k_2}^{b_2} \left( 2^{-k_2+2} u_1 + \frac{x_2+y_2}{2}, y_2 \right), \\ \phi_3(v_1) &= 2^{-k_1n_1} d_{k_1}^{\tilde{b}_1} \left( x_1, 2^{-k_1+2} v_1 + \frac{x_1+y_1}{2} \right), & \text{and} & \quad \phi_4(v_2) = 2^{-k_2n_2} d_{k_2}^{\tilde{b}_2} \left( x_2, 2^{-k_2+2} v_2 + \frac{x_2+y_2}{2} \right). \end{aligned}$$

It is not hard to verify that  $2^{k_1n_1} \phi_1^{\frac{x_1+y_1}{2}, 2^{-k_1+2}}(u_1) = s_{k_1}^{b_1}(u_1, y_1)$  for  $u_1 \in \mathbb{R}^{n_1}$  and likewise for the other three terms. This completes the proof of (3.1) when both  $x_1, y_1$  and  $x_2, y_2$  are close. Now fix  $x, y \in \mathbb{R}^n$  such that  $|x_1 - y_1| > 2^{-k_1+2}$  and  $|x_2 - y_2| > 2^{-k_2+2}$ . It follows that

$$\text{supp}(s_{k_1}^{b_1}(\cdot, y_1)) \cap \text{supp}(d_{k_1}^{\tilde{b}_1}(x_1, \cdot)) = \text{supp}(s_{k_2}^{b_2}(\cdot, y_2)) \cap \text{supp}(d_{k_2}^{\tilde{b}_2}(x_2, \cdot)) = \emptyset.$$

Then we can use the kernel representation of  $T$  to write

$$\begin{aligned} |\theta_{\vec{s}}(x, y)| &= \left| \int_{\mathbb{R}^{2n}} K(u, v) s_{k_1}^{b_1}(v_1, y_1) d_{k_1}^{\tilde{b}_1}(x_1, u_1) s_{k_2}^{b_2}(v_2, y_2) d_{k_2}^{\tilde{b}_2}(x_2, u_2) \tilde{b}(u) b(v) du dv \right| \\ &\lesssim \int_{\mathbb{R}^{2n}} |K(u, v) - K(x_1, u_2, v_1, v_2) - K(u_1, x_2, v_1, v_2) + K(x_1, x_2, v_1, v_2)| \\ &\quad \times |s_{k_1}^{b_1}(v_1, y_1) d_{k_1}^{\tilde{b}_1}(x_1, u_1) s_{k_2}^{b_2}(v_2, y_2) d_{k_2}^{\tilde{b}_2}(x_2, u_2)| du dv \\ &\leq \int_{|y_i-v_i|<2^{-k_i}} \int_{|x_i-u_i|<2^{-k_i}} \frac{|x_1-u_1|^\gamma |x_2-u_2|^\gamma}{|x_1-v_1|^{n_1+\gamma} |x_2-v_2|^{n_2+\gamma}} 2^{2k_1n_1} 2^{2k_2n_2} du dv \\ &\leq \int_{|y_i-v_i|<2^{-k_i}} \int_{|x_i-u_i|<2^{-k_i}} \frac{2^{k_1(2n_1-\gamma)} 2^{k_2(2n_2-\gamma)}}{(|x_1-y_1|/2+2^{-k_1})^{n_1+\gamma} (|x_2-y_2|/2+2^{-k_2})^{n_2+\gamma}} du dv \\ &\lesssim \frac{2^{-\gamma k_1} 2^{-\gamma k_2}}{(|x_1-y_1|+2^{-k_1})^{n_1+\gamma} (|x_2-y_2|+2^{-k_2})^{n_2+\gamma}} = \Phi_{k_1}^{n_1+\gamma}(x_1 - y_1) \Phi_{k_2}^{n_2+\gamma}(x_2 - y_2). \end{aligned}$$

Fix  $x, y \in \mathbb{R}^n$  such that  $|x_1 - y_1| \leq 2^{-k_1+2}$  and  $|x_2 - y_2| > 2^{-k_2+2}$ . Then we can write

$$\begin{aligned} |\theta_{\vec{s}}(x, y)| &= \left| \left\langle M_{\vec{b}} T M_b \left( s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2) \right), d_{k_1}^{\vec{b}_1}(x_1, \cdot) \otimes d_{k_2}^{\vec{b}_2}(x_2, \cdot) \right\rangle \right| \\ &= 2^{2k_1 n_1} 2^{2k_2 n_2} \left| \left\langle M_{\vec{b}} T M_b \left( \tilde{\phi}_1^{y_1, 2^{-k_1}} \otimes \phi_2^{\frac{x_2+y_2}{2}, 2^{-k_2+2}} \right), \tilde{\phi}_3^{x_1, 2^{-k_1}} \otimes \phi_4^{\frac{x_2+y_2}{2}, 2^{-k_2+2}} \right\rangle \right|, \end{aligned}$$

where

$$\tilde{\phi}_1(u_1) = 2^{-k_1 n_1} s_{k_1}^{b_1}(2^{-k} u_1 + y_1, y_1) \quad \text{and} \quad \tilde{\phi}_3(v_1) = 2^{-k_1 n_1} d_{k_1}^{\vec{b}_1}(x_1, 2^{-k} v_1 + x_1)$$

again are normalized bumps of order  $m$  (up to a constant multiple independent of  $x, y$ , and  $\vec{k}$ ). Since  $|x_2 - y_2| > 4 \cdot 2^{-k_2}$ , we can apply (4.10) to obtain the following estimate.

$$\begin{aligned} |\theta_{\vec{k}}(x, y)| &\lesssim 2^{2k_1 n_1} 2^{2k_2 n_2} \left( \frac{2^{-k_1 n_1} 2^{-k_2 n_2}}{(2^{k_2} |x_2 - y_2|)^{n_2 + \gamma}} \right) \\ &\lesssim \frac{2^{k_1 n_1} 2^{k_2 n_2}}{(1 + 2^{k_2} |x_2 - y_2|)^{n_2 + \gamma}} \lesssim \Phi_{k_1}^{n_1 + \gamma}(x_1 - y_1) \Phi_{k_2}^{n_2 + \gamma}(x_2 - y_2). \end{aligned}$$

A similar argument using (4.8) proves that (3.1) holds when  $|x_1 - y_1| > 2^{-k_1+2}$  and  $|x_2 - y_2| \leq 2^{-k_2+2}$ . This verifies that  $\theta_{\vec{k}}$  satisfies condition (3.1) for all  $x, y \in \mathbb{R}^n$ . Now to verify (3.2), recall that for  $W \in (C_0^\infty(\mathbb{R}^n))'$ ,  $f \in C_0^\infty(\mathbb{R}^n)$ , and  $x \in \mathbb{R}^n$ ,  $F(x) = \langle W, f^x \rangle$  is a differentiable function where  $\partial_{x_i} F(x) = \langle W, (\partial_{x_i} f)^x \rangle$ . Then  $\theta_{\vec{k}}$  is differentiable, and we can estimate

$$\begin{aligned} |\nabla_{x_1} \theta_{\vec{k}}(x, y)|^2 &= \sum_{j=1}^{n_1} \left| \left\langle M_{\vec{b}} T M_b (s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)), \partial_{x_{1,j}} (d_{k_1}^{\vec{b}_1}(x_1, \cdot)) \otimes d_{k_2}^{\vec{b}_2}(x_2, \cdot) \right\rangle \right|^2 \\ &\lesssim 2^{2k_1(n_1+1)} 2^{2k_2 n_2}, \end{aligned}$$

since  $2^{-k_1(n_1+1)} \partial_{x_{1,j}} (d_{k_1}^{\vec{b}_1}(x_1, \cdot))$  is again a normalized bump for  $x_1 = (x_{1,1}, \dots, x_{1,n_1}) \in \mathbb{R}^{n_1}$  (up to a constant multiple independent of  $x, y$ , and  $\vec{k}$ ). Therefore

$$|\theta_{\vec{k}}(x, y) - \theta_{\vec{k}}(x', y)| \leq \|\nabla_{x_1} \theta_{\vec{k}}(x, y)\|_{L^\infty} |x_1 - x'_1| \lesssim 2^{k_1 n_1} 2^{k_2 n_2} (2^{k_1} |x_1 - x'_1|).$$

This proves that  $\theta_{\vec{k}}$  verifies (3.2) via the equivalence in Remark 3.1. By the same argument, it follows that  $\theta_{\vec{k}}$  verifies (3.3)-(3.5). Now by the continuity of  $T$  from  $bC_0^\delta(\mathbb{R}^n)$  into  $(\tilde{b}C_0^\delta(\mathbb{R}^n))'$ , we have that

$$\int_{\mathbb{R}^{n_1}} \theta_{\vec{k}}(x, y) \tilde{b}_1(x_1) dx_1 = \lim_{R \rightarrow \infty} \left\langle M_{\vec{b}} T M_b (s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)), \lambda_{R, k_1} \otimes d_{k_2}^{\vec{b}_2}(x_2, \cdot) \right\rangle$$

where

$$\lambda_{R, k_1}(u_1) = \int_{|x_1| \leq R} d_{k_1}^{\vec{b}_1}(x_1, u_1) \tilde{b}_1(x_1) dx_1.$$

Note that for  $|u_1| > R + 2^{-k_1}$ , we have  $|u_1 - x_1| \geq |u_1| - |x_1| > 2^{-k_1}$  and hence  $\lambda_{R, s_1}(u_1) = 0$  for such  $u_1$ . Also for  $|u_1| < R - 2^{-k_1}$  and  $x \in \text{supp}(d_{k_1}^{\vec{b}_1}(\cdot, u_1))$ , it follows that  $|x_1| \leq |u_1| + |u_1 - x_1| < R$ . Since  $D_{k_1}^{\vec{b}_1} \tilde{b}_1 = 0$ ,  $\lambda_{R, s_1}(u_1) = 0$  for  $|u_1| < R - 2^{-k_1}$ . That is  $\text{supp}(\lambda_{R, s_1}) \subset B(0, R + 2^{-k_1}) \setminus B(0, R - 2^{-k_1})$ . Now take  $R > |y_1| + 2^{-k_1+1}$  so that  $\lambda_{R, k_1}$  and  $s_{k_1}^{b_1}(\cdot, y_1)$  have disjoint support. Now we split into two cases: (1) where  $|x_2 - y_2| \leq 2^{-k_1+2}$  and (2) where  $|x_2 - y_2| > 2^{-k_2+2}$ .

Case 1: ( $|x_2 - y_2| \leq 2^{-k_1+2}$ ) Here we take  $R > 2^{-k_1+6} + 2|y_1|$ . Consider

$$\mathcal{B} = \{B(u_1, 2^{-k_1}) : u_1 \in \text{supp}(\lambda_{R, k_1})\},$$

which is an open cover of  $\text{supp}(\lambda_{R, k_1})$ . Then by Vitali's covering lemma, there exists finite collection  $\{B_1, \dots, B_J\} \subset \mathcal{B}$  of disjoint balls such that  $\{3B_1, \dots, 3B_J\}$  forms an open cover of  $\text{supp}(\lambda_{R, k_1})$ . Let  $c_j \in \mathbb{R}^{n_1}$  be the center of  $B_j$  for each  $j = 1, \dots, J$ . Fix  $\chi \in C_0^\infty(\mathbb{R}^{n_1})$  such that  $\chi = 1$  on  $B(0, 1)$  and  $\text{supp}(\chi) \subset B(0, 2)$ . Let  $\tilde{\chi}_j(u_1) = \chi\left(\frac{u_1 - c_j}{3 \cdot 2^{-k_1}}\right)$ , and it follows

that  $\tilde{\chi}_j = 1$  on  $3B_j$  and  $\tilde{\chi}_j$  is supported inside  $6B_j$ . Finally define the partition of unity for  $3B_1 \cup \dots \cup 3B_J$ ,

$$\chi_j(u_1) = \frac{\tilde{\chi}_j(u_1)}{\sum_{k=1}^J \tilde{\chi}_k(u_1)} \quad \text{for } j = 1, \dots, J.$$

Let  $m \in \mathbb{N}_0$  be the integer specified by the weak boundedness and mixed weak boundedness properties for  $M_b T M_b$ . It follows that

$$\eta_j(u_1) = \frac{1}{\max_{|\alpha| \leq m} \|\partial^\alpha(\lambda_{R,k_1} \chi_j)\|_{L^\infty}} \chi_j(2^{-k_1+3} u_1 + c_j) \lambda_{R,k_1}(2^{-k_1+3} u_1 + c_j)$$

is a normalized bump of order  $m$  for each  $j = 1, \dots, J$ . Note that for each  $\beta \in \mathbb{N}_0^{n_1}$  with  $|\beta| \leq |\alpha| \leq m$

$$\begin{aligned} |\partial^\beta \lambda_{R,k_1}(u_1)| &\leq \int_{|x_1| \leq R} |\partial_{u_1}^\beta d_{k_1}^{\tilde{b}_1}(x_1, u_1) \tilde{b}_1(x_1)| dx_1 \\ &\leq 2^{k_1|\beta|} \int_{\mathbb{R}^{n_1}} |\partial_{u_1}^\beta d_{k_1}^{\tilde{b}_1}(x_1, u_1) \tilde{b}_1(x_1)| dx_1 \lesssim 2^{k_1|\beta|}. \end{aligned}$$

The importance here is that this estimate does not depend on  $R$ ; it does depend on  $k_1$  and  $\beta$ , but since we are taking a limit in  $R$  for a fixed  $k_1$  and  $|\beta| \leq m$ , this is not of consequence. Likewise for  $|\beta| \leq |\alpha| \leq m$  and  $u \in \text{supp}(\lambda_{R,k_1}) \cap 3B_j$

$$|\partial^\beta \chi_j(u)| = \left| \partial^\beta \left[ \frac{\tilde{\chi} \left( 3 \frac{u_1 - c_j}{2^{-k_1}} \right)}{\sum_{k=1}^J \tilde{\chi}_k \left( 3 \frac{u_1 - c_j}{2^{-k_1}} \right)} \right] \right| = 3^{|\beta|} 2^{|\beta|k_1} \left\| \partial^\beta \left[ \frac{\tilde{\chi}}{\sum_{k=1}^J \tilde{\chi}_k} \right] \right\|_{L^\infty(B(0,1))} \leq A_\beta 2^{|\beta|k_1},$$

for some constant  $A_\beta > 0$  depending only on  $\beta \in \mathbb{N}_0^{n_1}$ . Note that we use  $\tilde{\chi}_j \in C_0^\infty(\mathbb{R}^{n_1})$  and  $\sum_{k=1}^J \tilde{\chi}_k \geq 1$  on  $\text{supp}(\lambda_{R,k_1}) \cap 3B_j$ . Again the importance here is that this estimate does not depend on  $R$ ; it does depend on  $k_1$ ,  $\beta$ , and derivatives of  $\chi$ , but that is not a problem. Also define  $\phi(u_1) = 2^{-k_1 n_1} s_{k_1}^{b_1}(2^{-k_1+3} u_1 + y_1, y_1)$ , and it follows that  $\phi$  is a normalized bump up to a constant multiple. We now use that

$$\begin{aligned} \sum_{j=1}^J \max_{|\alpha| \leq m} \|\partial^\alpha(\lambda_{R,k_1} \chi_j)\|_{L^\infty} \eta_j^{c_j, 2^{-k_1+3}}(u_1) &= \sum_{j=1}^J \chi_j(u_1) \lambda_{R,k_1}(u_1) = \lambda_{R,k_1}(u_1), \\ \phi^{y_1, 2^{-k_1+3}}(u_1) &= 2^{-k_1 n_1} s_{k_1}^{b_1} \left( 2^{-k_1+3} \frac{u_1 - c_j}{2^{-k_1+3}} + y_1, y_1 \right) = 2^{-k_1 n_1} s_{k_1}^{b_1}(u_1, y_1), \end{aligned}$$

and since  $R > 2^{-k_1+6} + 2|y_1|$ , it follows that

$$|c_j - y_1| \geq |c_j| - |y_1| \geq R - 2^{-k_1} - |y_1| > 2^{-k_1+6} - 2^{-k_1} \geq 4 \cdot 2^{-k_1+3}.$$

Then we can apply (4.7) in the following way

$$\begin{aligned} &\left| \left\langle M_b T M_b(s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)), \lambda_{R,k_1} \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot) \right\rangle \right| \\ &\leq \sum_{j=1}^J \max_{|\alpha| \leq m} \|\partial^\alpha(\lambda_{R,k_1} \chi_j)\|_{L^\infty} \left| \left\langle T(\phi^{y_1, 2^{-k_1+3}} \otimes s_{k_2}(\cdot, y_2)), \eta_j^{c_j, 2^{-k_1+3}} \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot) \right\rangle \right| \\ &\leq \sum_{j=1}^J A_{k_1, m} \frac{2^{k_2 n_2} 2^{-k_1 n_1}}{(2^{k_1} |y_1 - c_j|)^{n_1}} \lesssim \sum_{j=1}^J A_{k_1, m} \frac{2^{k_2 n_2} 2^{-2k_1 n_1}}{R^{n_1}} = A_{k_1, m} \frac{2^{k_2 n_2} 2^{-2k_1 n_1}}{R^{n_1}} J, \\ &\text{where } A_{k_1, m} = \max_{|\beta|+|\gamma| \leq m} 2^{k_1(|\beta|+|\gamma|)} A_\gamma. \end{aligned}$$

Now we use that  $B_1, \dots, B_J$  is a disjoint collection of open sets to estimate  $J$ :

$$\begin{aligned} J &\lesssim 2^{-k_1 n_1} \sum_{j=1}^J |B_j| = 2^{-k_1 n_1} \left| \bigcup_{j=1}^J B_j \right| \leq 2^{-k_1 n_1} |B(0, R + 2^{-k_1+3}) \setminus B(0, R - 2^{-k_1+3})| \\ &\lesssim 2^{-k_1(n_1+1)} R^{n_1-1}. \end{aligned}$$

Note that each  $B_j \subset B(0, R + 2^{-k_1+3}) \setminus B(0, R - 2^{-k_1+3})$  since  $c_j \in \text{supp}(\lambda_{R,k_1}) \subset B(0, R + 2^{-k_1+3}) \setminus B(0, R - 2^{-k_1+3})$  and each  $B_j$  has radius  $2^{-k_1}$ . Therefore

$$\begin{aligned} & \left| \left\langle M_{\tilde{b}} T M_b(s_{k_1}(\cdot, y_1) \otimes s_{k_2}(\cdot, y_2)), \lambda_{R,k_1} \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot) \right\rangle \right| \\ & \lesssim A_{k_1,m} \frac{2^{-k_1(2n_1+\gamma)} 2^{k_2 n_2}}{R^{n_1}} 2^{-k_1(n_1+1)} R^{n_1-1} = A_{k_1,m} \frac{2^{-k_1(n_1-1)} 2^{k_2 n_2}}{R}, \end{aligned}$$

which tends to zero as  $R \rightarrow \infty$ . This completes the proof for the first case.

Case 2: ( $|x_2 - t_2| > 2^{-k_2+2}$ ) Since  $\lambda_{R,k_1}$  and  $s_{k_1}(\cdot, y_1)$  have disjoint support, we can use the full kernel representation for  $T$  to compute

$$\begin{aligned} & \left| \left\langle M_{\tilde{b}} T M_b(s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)), \lambda_{R,k_1} \otimes d_{k_2}^{\tilde{b}_2}(x_1, \cdot) \right\rangle \right| \\ & = \left| \iint_{\mathbb{R}^{2n}} K(u, v) s_{k_1}^{b_1}(v_1, y_1) s_{k_2}^{b_2}(v_2, y_2) \lambda_{R,k_1}(u_1) d_{k_2}^{\tilde{b}_2}(x_2, u_2) \tilde{b}(u) b(v) du dv \right| \\ & \lesssim \iint_{\mathbb{R}^{2n}} \frac{1}{|u_1 - v_1|^{n_1} |u_2 - v_2|^{n_2}} |s_{k_1}^{b_1}(v_1, y_1) s_{k_2}^{b_2}(v_2, y_2) \lambda_{R,k_1}(u_1) d_{k_2}^{\tilde{b}_2}(x_2, u_2)| du dv \\ & \lesssim \iint_{\mathbb{R}^{2n}} \frac{2^{k_2 n_2}}{(|u_1| - |t_1| - |t_1 - v_1|)^{n_1}} |s_{k_1}^{b_1}(v_1, y_1) s_{k_2}^{b_2}(v_2, y_2) \lambda_{R,k_1}(u_1) d_{k_2}^{\tilde{b}_2}(x_2, u_2)| du dv \\ & \lesssim 2^{k_2 n_2} R^{-n_1} \int_{\mathbb{R}^{n_1}} |\lambda_{R,k_1}(u_1)| du_1 \lesssim 2^{k_2 n_2} 2^{-k_1} R^{-1}, \end{aligned}$$

which again tends to zero as  $R \rightarrow \infty$ . Therefore  $\theta_{\tilde{k}}$  has integral zero in  $x_1$ , and a similar argument proves that it has integral zero in  $x_2$  as well.  $\square$

By symmetry, it follows that each of the following define collections of biparameter Littlewood-Paley-Stein operators:

$$\begin{aligned} \theta_k^2(x, y) &= \left\langle M_{\tilde{b}} T M_b(s_{k_1}^{b_1}(\cdot, y_1) \otimes d_{k_2}^{b_2}(\cdot, y_2)), d_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes s_{k_2}^{\tilde{b}_2}(x_2, \cdot) \right\rangle, \\ \theta_k^3(x, y) &= \left\langle M_{\tilde{b}} T M_b(d_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)), s_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot) \right\rangle, \text{ and} \\ \theta_k^4(x, y) &= \left\langle M_{\tilde{b}} T M_b(d_{k_1}^{b_1}(\cdot, y_1) \otimes d_{k_2}^{b_2}(\cdot, y_2)), s_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes s_{k_2}^{\tilde{b}_2}(x_2, \cdot) \right\rangle. \end{aligned}$$

Furthermore, these kernels satisfy

$$\begin{aligned} \int_{\mathbb{R}^{n_1}} \theta_k^2(x, y) \tilde{b}_1(x_1) dx_1 &= \int_{\mathbb{R}^{n_2}} \theta_k^2(x, y) b_2(y_2) dy_2 = 0, \\ \int_{\mathbb{R}^{n_1}} \theta_k^2(x, y) b_1(y_1) dy_1 &= \int_{\mathbb{R}^{n_2}} \theta_k^2(x, y) \tilde{b}_2(x_2) dx_2 = 0, \text{ and} \\ \int_{\mathbb{R}^{n_1}} \theta_k^2(x, y) b_1(y_1) dy_1 &= \int_{\mathbb{R}^{n_2}} \theta_k^2(x, y) b_2(y_2) dy_2 = 0. \end{aligned}$$

**Definition 4.4.** A biparameter singular integral operator satisfies the biparameter  $Tb = T^* \tilde{b} = 0$  condition if the following two conditions hold: (1) Let  $\psi_1 \in C_0^\infty(\mathbb{R}^{n_1})$ ,  $\psi_2, \varphi_2 \in C_0^\infty(\mathbb{R}^{n_2})$ , and  $\eta_R \in C_0^\infty(\mathbb{R}^{n_1})$  such that  $\eta_R = 1$  on  $B_1(0, R) \subset \mathbb{R}^{n_1}$  and  $\text{supp}(\eta_R) \subset B_1(0, 2R) \subset \mathbb{R}^{n_1}$ . If  $b_1 \psi_1$  has mean zero and either  $b_2 \varphi_2$  or  $b_2 \psi_2$  has mean zero, then

$$(4.12) \quad \langle T(b_1 \otimes b_2 \psi_2), \tilde{b}_1 \psi_1 \otimes \tilde{b}_2 \varphi_2 \rangle := \lim_{R \rightarrow \infty} \langle M_{\tilde{b}} T M_b(\eta_R \otimes \psi_2), \psi_1 \otimes \varphi_2 \rangle = 0,$$

$$(4.13) \quad \langle T(b_1 \psi_1 \otimes b_2 \psi_2), \tilde{b}_1 \otimes \tilde{b}_2 \varphi_2 \rangle := \lim_{R \rightarrow \infty} \langle M_{\tilde{b}} T M_b(\psi_1 \otimes \psi_2), \eta_R \otimes \varphi_2 \rangle = 0,$$



and (2) let  $\psi_2 \in C_0^\infty(\mathbb{R}^{n_2})$ ,  $\psi_1, \phi_1 \in C_0^\infty(\mathbb{R}^{n_1})$ , and  $\eta_R \in C_0^\infty(\mathbb{R}^{n_2})$  such that  $\eta_R = 1$  on  $B_2(0, R) \subset \mathbb{R}^{n_1}$  and  $\text{supp}(\eta_R) \subset B_2(0, 2R) \subset \mathbb{R}^{n_2}$ . If  $b_2\psi_2$  has mean zero and either  $b_1\phi_1$  or  $b_1\psi_1$  has mean zero, then

$$\begin{aligned} \langle T(b_1\psi_1 \otimes b_2), \tilde{b}_1\phi_1 \otimes \tilde{b}_2\psi_2 \rangle &:= \lim_{R \rightarrow \infty} \langle M_{\tilde{b}}TM_b(\psi_1 \otimes \eta_R), \phi_1 \otimes \psi_2 \rangle = 0, \\ \langle T(b_1\psi_1 \otimes b_2\psi_2), \tilde{b}_1\phi_1 \otimes \tilde{b}_2 \rangle &:= \lim_{R \rightarrow \infty} \langle M_{\tilde{b}}TM_b(\psi_1 \otimes \psi_2), \phi_1 \otimes \eta_R \rangle = 0. \end{aligned}$$

Next we prove Theorem 4.

*Proof.* Let  $S_k^b = S_{k_1}^{b_1} \otimes S_{k_2}^{b_2}$  and  $S_k^{\tilde{b}} = S_{k_1}^{\tilde{b}_1} S_{k_2}^{\tilde{b}_2}$ , where  $S_{k_1}^{b_1}$ ,  $S_{k_2}^{b_2}$ ,  $S_{k_1}^{\tilde{b}_1}$ , and  $S_{k_2}^{\tilde{b}_2}$  be the approximations to identity with respect to  $b_1$  and  $b_2$  respectively constructed in (3.7). Also define  $D_{k_1}^{b_1} = S_{k_1+1}^{b_1} - S_{k_1}^{b_1}$ ,  $D_{k_2}^{b_2} = S_{k_2+1}^{b_2} - S_{k_2}^{b_2}$ ,  $D_{k_1}^{\tilde{b}_1} = S_{k_1+1}^{\tilde{b}_1} - S_{k_1}^{\tilde{b}_1}$ ,  $D_{k_2}^{\tilde{b}_2} = S_{k_2+1}^{\tilde{b}_2} - S_{k_2}^{\tilde{b}_2}$ ,  $D_k^b = D_{k_1}^{b_1} D_{k_2}^{b_2}$ , and  $D_k^{\tilde{b}} = D_{k_1}^{\tilde{b}_1} D_{k_2}^{\tilde{b}_2}$ . It follows that  $M_{b_j} S_{k_j}^{b_j} M_{b_j} f_j \rightarrow b_j f_j$  and  $M_{b_j} S_{-k_j}^{b_j} M_{b_j} f_j \rightarrow 0$  in  $b_j C_0^\delta(\mathbb{R}^{n_j})$  as  $k_j \rightarrow \infty$  for  $j = 1, 2$ , whenever  $f_j \in C_0^{0,1}(\mathbb{R}^{n_j})$  and

$$\int_{\mathbb{R}^{n_j}} f_j(x_j) b_j(x_j) dx_j = 0.$$

This was proved originally in [DJS85], and the proof is also available in [Har13a]. It follows that  $M_{b_j} S_{k_j}^{b_j} M_{b_j} f \rightarrow b f$  and  $M_{b_j} S_{-k_j}^{b_j} M_{b_j} f \rightarrow 0$  in  $b C_0^\delta(\mathbb{R}^n)$  as  $k_j \rightarrow \infty$  for  $j = 1, 2$ , whenever  $f \in C_0^{0,1}(\mathbb{R}^n)$  and

$$\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = 0.$$

Let  $f, g \in C_0^{0,1}(\mathbb{R}^n)$  such that

$$\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = \int_{\mathbb{R}^{n_1}} g(x) \tilde{b}(x) dx_1 = \int_{\mathbb{R}^{n_2}} g(x) \tilde{b}(x) dx_2 = 0.$$

Then by the continuity of  $T$  from  $b C_0^\delta(\mathbb{R}^n)$  into  $(\tilde{b} C_0^\delta(\mathbb{R}^n))'$ ,

$$\begin{aligned} \langle M_{\tilde{b}} T M_b f, g \rangle &= \lim_{N_2 \rightarrow \infty} \langle M_{\tilde{b}_2} T M_{b_2} S_{N_2}^{b_2} M_{b_2} f, S_{N_2}^{\tilde{b}_2} M_{\tilde{b}_2} g \rangle - \langle M_{\tilde{b}_2} T M_{b_2} S_{-N_2}^{b_2} M_{b_2} f, S_{-N_2}^{\tilde{b}_2} M_{\tilde{b}_2} g \rangle \\ &= \sum_{k_2 \in \mathbb{Z}} \langle M_{\tilde{b}_2} T M_{b_2} S_{k_2+1}^{b_2} M_{b_2} f, D_{k_2}^{\tilde{b}_2} M_{\tilde{b}_2} g \rangle - \langle M_{\tilde{b}_2} T M_{b_2} D_{k_2}^{b_2} M_{b_2} f, S_{k_2}^{\tilde{b}_2} M_{\tilde{b}_2} g \rangle \\ &= \sum_{k_2 \in \mathbb{Z}} \lim_{N_1 \rightarrow \infty} \langle M_{\tilde{b}} T M_b S_{k_2+1}^{b_2} S_{N_1}^{b_1} M_{b_1} f, D_{k_2}^{\tilde{b}_2} S_{N_1}^{\tilde{b}_1} M_{\tilde{b}_2} g \rangle + \langle M_{\tilde{b}} T M_b D_{k_2}^{b_2} S_{N_1}^{b_1} M_{b_1} f, S_{k_2}^{\tilde{b}_2} S_{N_1}^{\tilde{b}_1} M_{\tilde{b}_2} g \rangle \\ &\quad - \langle M_{\tilde{b}} T M_b S_{k_2+1}^{b_2} S_{-N_1}^{b_1} M_{b_1} f, D_{k_2}^{\tilde{b}_2} S_{-N_1}^{\tilde{b}_1} M_{\tilde{b}_2} g \rangle - \langle M_{\tilde{b}} T M_b D_{k_2}^{b_2} S_{-N_1}^{b_1} M_{b_1} f, S_{k_2}^{\tilde{b}_2} S_{-N_1}^{\tilde{b}_1} M_{\tilde{b}_2} g \rangle \\ &= \sum_{k_1, k_2 \in \mathbb{Z}} \langle M_{\tilde{b}} T M_b S_{k_2+1}^{b_2} S_{k_1+1}^{b_1} M_{b_1} f, D_{k_2}^{\tilde{b}_2} D_{k_1}^{\tilde{b}_1} M_{\tilde{b}_2} g \rangle + \langle M_{\tilde{b}} T M_b D_{k_2}^{b_2} S_{k_1+1}^{b_1} M_{b_1} f, S_{k_2}^{\tilde{b}_2} D_{k_1}^{\tilde{b}_1} M_{\tilde{b}_2} g \rangle \\ &\quad + \langle M_{\tilde{b}} T M_b S_{k_2+1}^{b_2} D_{k_1}^{\tilde{b}_1} M_{b_1} f, D_{k_2}^{\tilde{b}_2} S_{k_1}^{\tilde{b}_1} M_{\tilde{b}_2} g \rangle + \langle M_{\tilde{b}} T M_b D_{k_2}^{b_2} D_{k_1}^{b_1} M_{b_1} f, S_{k_2}^{\tilde{b}_2} S_{k_1}^{\tilde{b}_1} M_{\tilde{b}_2} g \rangle \\ &= \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{j=1}^4 \langle \Theta_k^j M_{b_1} f, M_{\tilde{b}_2} g \rangle \end{aligned}$$

where  $\Theta_j$  for  $j = 1, 2, 3, 4$  are defined as follows with their respective kernels

$$\begin{aligned} \Theta_k^1 &= D_k^{\tilde{b}} M_{\tilde{b}} T M_b S_{k+1}^b; & \Theta_k^1(x, y) &= \langle M_{\tilde{b}} T M_b (S_{k_1+1}^{b_1}(\cdot, y_1) \otimes S_{k_2+1}^{b_2}(\cdot, y_2)), d_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot) \rangle, \\ \Theta_k^2 &= D_{k_1}^{\tilde{b}_1} S_{k_2}^{\tilde{b}_2} M_{\tilde{b}} T M_b S_{k_1+1}^{b_1} D_{k_2}^{b_2}; & \Theta_k^2(x, y) &= \langle M_{\tilde{b}} T M_b (S_{k_1+1}^{b_1}(\cdot, y_1) \otimes d_{k_2}^{b_2}(\cdot, y_2)), d_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes S_{k_2}^{\tilde{b}_2}(x_2, \cdot) \rangle, \\ \Theta_k^3 &= S_{k_1}^{\tilde{b}_1} D_{k_2}^{\tilde{b}_2} M_{\tilde{b}} T M_b D_{k_1}^{b_1} S_{k_2+1}^{b_2}; & \Theta_k^3(x, y) &= \langle M_{\tilde{b}} T M_b (d_{k_1}^{b_1}(\cdot, y_1) \otimes S_{k_2+1}^{b_2}(\cdot, y_2)), S_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot) \rangle, \\ \Theta_k^4 &= S_{k_1}^{\tilde{b}_1} M_{\tilde{b}} T M_b D_{k_1}^{b_1} D_{k_2}^{b_2}; & \Theta_k^4(x, y) &= \langle M_{\tilde{b}} T M_b (d_{k_1}^{b_1}(\cdot, y_1) \otimes d_{k_2}^{b_2}(\cdot, y_2)), S_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes S_{k_2}^{\tilde{b}_2}(x_2, \cdot) \rangle. \end{aligned}$$

By Lemma 4.1,  $\theta_s^1$  satisfies (3.1)-(3.5) and

$$\int_{\mathbb{R}^{n_1}} \theta_k^1(x, y) b_1(x_1) dx_1 = \int_{\mathbb{R}^{n_2}} \theta_k^1(x, y) b_2(x_2) dx_2 = 0.$$

By the biparameter  $Tb = T^*b = 0$  assumption on  $T$ , we also have

$$\int_{\mathbb{R}^{n_1}} \theta_k^1(x, y) b_1(y_1) dy_1 = \int_{\mathbb{R}^{n_2}} \theta_k^1(x, y) b_2(y_2) dy_2 = 0.$$

Then by Theorem (3.1),

$$\sum_{\vec{k} \in \mathbb{Z}^2} \left| \int_{\mathbb{R}^n} \theta_k^1 f(x) g(x) dx \right| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

The same holds for  $\Theta_s^j$  when  $j = 2, 3, 4$ , and so it follows that

$$|\langle Tf, g \rangle| \leq \sum_{j=1}^4 \sum_{\vec{k} \in \mathbb{Z}^2} \left| \int_{\mathbb{R}^n} \Theta_k^j f(x) g(x) dx \right| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Therefore by density,  $T$  can be extended to a bounded operator on  $L^p$  for  $1 < p < \infty$ .  $\square$

## 5. PROOF OF BOUNDS FOR $C_\Gamma, \tilde{C}_\Gamma$ AND $C_\Gamma^*$

In this section, we use Theorem 4 to prove bounds for  $C_\Gamma$ , its parameterized version  $\tilde{C}_\Gamma$ , which we define now, and the maximal operator  $C_\Gamma^*$ .

For appropriate  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , define

$$\tilde{C}_\Gamma M_b f(x) = \lim_{t_1, t_2 \rightarrow 0^+} \int_{\mathbb{R}^2} \frac{\gamma_1(x_1) - \gamma_1(y_1)}{(\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2} \frac{\gamma_2(x_2) - \gamma_2(y_2)}{(\gamma_2(x_2) - \gamma_2(y_2))^2 + t_2^2} f(y) b(y) dy,$$

where  $b(y) = \gamma'_1(y_1) \gamma'_2(y_2)$ . We call this the parameterized version of  $C_\Gamma$  since

$$\tilde{C}_\Gamma M_b f(x) = C_\Gamma(f \circ \gamma^{-1})(\gamma(x)),$$

and furthermore, the  $L^p(\Gamma)$  bound for  $C_\Gamma$  can be reduced to  $L^p(\mathbb{R}^2)$  bounds for  $\tilde{C}_\Gamma$  via (2.1). It is not hard to see that the kernel of  $\tilde{C}_\Gamma$  is

$$\frac{1}{(\gamma_1(x_1) - \gamma_1(y_1))(\gamma_1(x_2) - \gamma_1(y_2))},$$

which is a biparameter Calderón-Zygmund kernel. In the next proposition, we prove that  $\tilde{C}_\Gamma f$  is well-defined for appropriate  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  and hence  $C_\Gamma g$  is also well defined for appropriate  $g : \Gamma \rightarrow \mathbb{C}$ . Define the complex log function with the negative real branch cut, that is for  $z \in \mathbb{C}$  we define

$$\log(z) = \ln(|z|) + i \text{Arg}(z),$$

where  $\ln : (0, \infty) \rightarrow \mathbb{R}$  logarithm base  $e$  function with positive real domain and  $\text{Arg}(z)$  is the principle argument of  $z$  taking values in  $(-\pi, \pi]$ . Note that for  $u \in (0, \infty)$ ,  $\ln(u) = \log(u)$ ; we use this notation to emphasize when the input is real versus complex.

**Proposition 5.1.** *Assume  $\Gamma$  satisfies the hypotheses of Theorem 1. For all  $f \in C_0^\infty(\mathbb{R}^2)$  and  $x \in \mathbb{R}^2$ ,*

$$\tilde{C}_\Gamma(bf)(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \partial_{y_1} \partial_{y_2} f(y) dy.$$

Also, for all  $f, g \in C_0^\infty(\mathbb{R}^2)$ , the pairing  $\langle \tilde{C}_\Gamma(bf), bg \rangle$  can be realized as any of the following absolutely convergent integrals:

$$\begin{aligned} & \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \partial_{y_1} \partial_{y_2} f(y) g(x) b(x) dy dx, \\ & \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) f(y) \partial_{x_1} \partial_{x_2} g(x) b(y) dy dx, \\ & - \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \partial_{y_1} f(y) \partial_{x_2} g(x) b(x_1, y_2) dy dx, \\ & - \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \partial_{y_2} f(y) \partial_{x_1} g(x) b(y_1, x_2) dy dx. \end{aligned}$$

*Proof.* We first note that for  $x_j, y_j \in \mathbb{R}$

$$\begin{aligned} q_{t_j}(\gamma_j(x_j) - \gamma_j(y_j)) \gamma_j'(y_j) &= \frac{1}{\pi} \frac{\gamma_j(x_j) - \gamma_j(y_j)}{(\gamma_j(x_j) - \gamma_j(y_j))^2 + t_j^2} \gamma_j'(y_j) \\ (5.1) \quad &= -\frac{1}{2\pi} \partial_{y_j} \log((\gamma_j(x_j) - \gamma_j(y_j))^2 + t_j^2). \end{aligned}$$

The derivative of  $\log$  is well defined here since we defined it with the negative real branch cut, and for all  $x_j, y_j \in \mathbb{R}$ , we have  $\operatorname{Re}((\gamma_j(x_j) - \gamma_j(y_j))^2 + t_j^2) \geq t_j^2 > 0$ . Now for  $f \in C_0^\infty(\mathbb{R}^2)$  and  $x \in \mathbb{R}^2$ , we compute the following pointwise limit

$$\begin{aligned} \tilde{C}_\Gamma(bf)(x) &= \lim_{t_1, t_2 \rightarrow 0^+} \int_{\mathbb{R}^2} q_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) q_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) f(y) \gamma_1'(y_1) \gamma_2'(y_2) dy \\ &= \lim_{t_1, t_2 \rightarrow 0^+} \int_{\mathbb{R}^2} \left[ -\frac{1}{2\pi} \partial_{y_1} \log((\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2) \right] \left[ -\frac{1}{2\pi} \partial_{y_2} \log((\gamma_2(x_2) - \gamma_2(y_2))^2 + t_2^2) \right] f(y) dy \\ &= \lim_{t_1, t_2 \rightarrow 0^+} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} [\log((\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2)] [\log((\gamma_2(x_2) - \gamma_2(y_2))^2 + t_2^2)] \partial_{y_1} \partial_{y_2} f(y) dy \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \partial_{y_1} \partial_{y_2} f(y) dy. \end{aligned}$$

We integrate by parts in  $y_1$  and  $y_2$  above, and the boundary terms vanish since  $f$  is compactly supported. Also to justify the last inequality, note the following holds for all  $x_j \neq y_j$ , so that we can apply dominated convergence: the following pointwise limit exists

$$\lim_{t_1, t_2 \rightarrow 0^+} \log((\gamma_j(x_j) - \gamma_j(y_j))^2 + t_j^2) \partial_{y_1} \partial_{y_2} f(y) = \log((\gamma_j(x_j) - \gamma_j(y_j))^2) \partial_{y_1} \partial_{y_2} f(y),$$

and the integrand is dominated by an integrable function independent of  $t_1, t_2 < 1/4$

$$|\log((\gamma_j(x_j) - \gamma_j(y_j))^2 + t_j^2)| \leq |\ln(|(\gamma_j(x_j) - \gamma_j(y_j))^2 + t_j^2|)| + \pi \lesssim |\ln((x_j - y_j)^2)| + 1.$$

Since  $\ln(|\cdot|)$  is locally integrable and  $f \in C_0^\infty(\mathbb{R}^2)$ , we may apply dominated convergence in the last line above. Now take  $f, g \in C_0^\infty(\mathbb{R}^2)$ , and it immediately follows that

$$\begin{aligned} \langle M_b \tilde{C}_\Gamma M_b f, g \rangle &= \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \\ &\quad \times \partial_{y_1} \partial_{y_2} f(y) g(x) \gamma_1'(x_1) \gamma_2'(x_2) dy dx. \end{aligned}$$

We also have that

$$\begin{aligned}
\langle M_b \tilde{C}_\Gamma M_b f, g \rangle &= \lim_{t_1, t_2 \rightarrow 0^+} \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2 + t_2^2) \\
&\quad \times \partial_{y_1} \partial_{y_2} f(y) g(x) \gamma_1'(x_1) \gamma_2'(x_2) dy dx \\
&= \lim_{t_1, t_2 \rightarrow 0^+} \int_{\mathbb{R}^4} q_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) q_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) f(y) g(x) \gamma_1'(y_1) \gamma_2'(y_2) \gamma_1'(x_1) \gamma_2'(x_2) dy dx \\
&= \lim_{t_1, t_2 \rightarrow 0^+} \frac{1}{4\pi^2} \int_{\mathbb{R}^4} [\partial_{x_1} \log((\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2)] \\
&\quad \times [-\partial_{y_2} \log((\gamma_2(x_2) - \gamma_2(y_2))^2 + t_2^2)] f(y) g(x) \gamma_1'(y_1) \gamma_2'(x_2) dy dx \\
&= \lim_{t_1, t_2 \rightarrow 0^+} -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2) \\
&\quad \times \log((\gamma_2(x_2) - \gamma_2(y_2))^2 + t_2^2) \partial_{y_2} f(y) \partial_{x_1} g(x) \gamma_1'(y_1) \gamma_2'(x_2) dy dx \\
&= -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \partial_{y_2} f(y) \partial_{x_1} g(x) \gamma_1'(y_1) \gamma_2'(x_2) dy dx.
\end{aligned}$$

Here we integrate by parts in  $x_1$  and  $y_2$  and use dominated convergence in essentially the same way as above. A similar argument verifies the other formulas for  $\langle \tilde{C}_\Gamma(bf), bg \rangle$ .  $\square$

Note that we cannot use properties of logs to replace the integrand above by

$$4 \log(\gamma_1(x_1) - \gamma_1(y_1)) \log(\gamma_2(x_2) - \gamma_2(y_2)).$$

This is because  $\operatorname{Re}[(\gamma_j(x_j) - \gamma_j(y_j))^2] > 0$  for  $x_j \neq y_j$ , and furthermore recall that we showed that  $\operatorname{Re}[(\gamma_j(x_j) - \gamma_j(y_j))^2] \geq (1 - \lambda_j^2)(x_j - y_j)^2$ . So this term avoids the branch cut of  $\log$ , but  $\operatorname{Re}[\gamma_j(x_j) - \gamma_j(y_j)]$  may change sign, which causes problems with the complex log function.

**Lemma 5.1.** *Suppose  $L_j : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz function with small Lipschitz constant  $\lambda_j < 1$  for  $j = 1, 2$ , and define  $\gamma(x) = (\gamma_1(x_1), \gamma_2(x_2)) = (x_1 + iL_1(x_1), x_2 + iL_2(x_2))$ . If  $\psi \in C_0^\infty(\mathbb{R})$  is a normalized bump of any order with mean zero, then*

$$\sup_{u_j \in \mathbb{R}, R_j > 0} \left| \int_{\mathbb{R}} \log((\gamma_j(x_j) - \gamma_j(y_j))^2) R_j^{-1} \psi\left(\frac{u_j - y_j}{R_j}\right) dy_j \right| \lesssim 1,$$

where the suppressed constant does not depend on  $\psi$ ,  $x_j$ , or  $\gamma$ . In other words,  $\log((\gamma_j(x_j) - \gamma_j(\cdot))^2) \in BMO(\mathbb{R})$  with norm independent of  $x_j$ , and  $\gamma$ . In particular this holds when  $\psi(u_j) = \phi'(u_j)$  for some normalized bump  $\phi \in C_0^\infty(\mathbb{R})$  of order at least 1.

*Proof.* Let  $\psi \in C_0^\infty(\mathbb{R})$  be a normalized bump with integral zero. For  $|u_j - x_j| \leq 2R_j$

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \log((\gamma_j(x_j) - \gamma_j(y_j))^2) R_j^{-1} \psi\left(\frac{u_j - y_j}{R_j}\right) dy_j \right| \\
&\leq \frac{\|\psi\|_{L^\infty}}{R_j} \int_{u_j - x_j - R_j}^{u_j - x_j + R_j} |\log((\gamma_j(x_j) - \gamma_j(x_j + y_j))^2) - \log(R_j^2)| dy_j \\
&\leq \int_{-3}^3 \left( \ln \left( \frac{|\gamma_j(x_j) - \gamma_j(x_j + R_j y_j)|^2}{R_j^2} \right) + \pi \right) dy_j \\
&\lesssim \int_{-3}^3 (1 + |\ln(|y_j|)|) dy_j \lesssim 1.
\end{aligned}$$

Here we use that for  $|y_j| \leq 3$

$$(1 - \lambda_j^2)|y_j|^2 \leq \frac{|\gamma_j(x_j) - \gamma_j(x_j + R_j y_j)|^2}{R_j^2} \leq (1 + \lambda_j^2)|y_j|^2 \leq 4|y_j|^2 \leq 36.$$

Now for  $|u_j - x_j| > 2R_j$ , we estimate as follows

$$\begin{aligned}
& \left| \int_{\mathbb{R}} \log((\gamma_j(x_j) - \gamma_j(y_j))^2) R_j^{-1} \psi\left(\frac{u_j - y_j}{R_j}\right) dy_j \right| \\
& \leq \frac{\|\psi\|_{L^\infty}}{R_j} \int_{u_j - x_j - R_j}^{u_j - x_j + R_j} \left| \log((\gamma_j(x_j) - \gamma_j(x_j + y_j))^2) - \log((\gamma_j(x_j) - \gamma_j(u_j))^2) \right| dy_j \\
& \lesssim 1 + \frac{1}{R_j} \int_{u_j - x_j - R_j}^{u_j - x_j + R_j} \left| \ln\left(\frac{|\gamma_j(x_j) - \gamma_j(x_j + y_j)|^2}{|\gamma_j(x_j) - \gamma_j(u_j)|^2}\right) \right| dy_j \\
& \lesssim 1 + \frac{1}{R_j} \int_{|y_j - (u_j - x_j)| < R_j} \left| \ln\left(\frac{|y_j|}{|u_j - x_j|}\right) \right| dy_j \\
& \leq 1 + \frac{1}{R_j} \int_{|y_j - (u_j - x_j)| < R_j} \left| \ln\left(\frac{|u_j - x_j| + |y_j - (u_j - x_j)|}{|u_j - x_j|}\right) \right| dy_j \\
& \quad + \frac{1}{R_j} \int_{|y_j - (u_j - x_j)| < R_j} \left| \ln\left(\frac{|u_j - x_j|}{|u_j - x_j| - |y_j - (u_j - x_j)|}\right) \right| dy_j \\
& \leq 1 + \frac{1}{R_j} \int_{|y_j - (u_j - x_j)| < R_j} (\ln(3/2) + \ln(2)) dy_j \lesssim 1.
\end{aligned}$$

This completes the proof.  $\square$

Now we prove that  $\tilde{C}_\Gamma$  satisfies the hypotheses of Theorem 4.

**Proposition 5.2.** *Assume  $\Gamma$  satisfies the hypotheses of Theorem 1. The operator  $M_b \tilde{C}_\Gamma M_b$  satisfies the weak boundedness and mixed weak boundedness properties, where  $b(x) = \gamma'_1(x_1) \gamma'_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^2$ .*

*Proof.* Let  $\varphi_j, \psi_j \in C_0^\infty$  be normalized bumps,  $x \in \mathbb{R}^2$ , and  $R_1, R_2 > 0$ . Then

$$\begin{aligned}
& \left| \left\langle M_b \tilde{C}_\Gamma M_b(\varphi_1^{x_1, R_1} \otimes \varphi_2^{x_2, R_2}), \psi_1^{x_1, R_1} \otimes \psi_2^{x_2, R_2} \right\rangle \right| \\
& = \frac{1}{4\pi^2} \left| \int_{\mathbb{R}^4} \log((\gamma_1(u_1) - \gamma_1(v_1))^2) \log((\gamma_2(u_2) - \gamma_2(v_2))^2) \right. \\
& \quad \left. \times (\varphi_1^{x_1, R_1})'(v_1) (\varphi_2^{x_2, R_2})'(v_2) \psi_1^{x_1, R_1}(u_1) \psi_2^{x_2, R_2}(u_2) du dv \right| \\
& \leq \frac{1}{4\pi^2} \int_{x_1 - R_1}^{x_1 + R_1} \int_{x_2 - R_2}^{x_2 + R_2} \left| \int_{\mathbb{R}^2} \log((\gamma_1(u_1) - \gamma_1(v_1))^2) \log((\gamma_2(u_2) - \gamma_2(v_2))^2) \right. \\
& \quad \left. \times R_1^{-1} (\varphi_1')^{x_1, R_1}(v_1) R_2^{-1} (\varphi_2')^{x_2, R_2}(v_2) dv \right| du \lesssim R_1 R_2.
\end{aligned}$$

The last inequality holds due to Lemma 5.1. Then  $\tilde{C}_\Gamma$  satisfies the weak boundedness property. Now we verify the mixed weak boundedness properties for  $\tilde{C}_\Gamma$ : we first verify (4.7). Let  $x_1 \in \mathbb{R}$ ,  $R_1 > 0$ , and  $\varphi_j, \psi_j \in C_0^\infty(\mathbb{R})$  be

normalized bumps. Then for  $x_1, x_2, y_2 \in \mathbb{R}$  and  $R_1, R_2 > 0$  such that  $|x_1 - y_1| > 4R_1$

$$\begin{aligned}
& \left| \left\langle M_b \tilde{C}_\Gamma M_b(\varphi_1^{y_1, R_1} \otimes \varphi_2^{x_2, R_2}), \psi_1^{x_1, R_1} \otimes \psi_2^{x_2, R_2} \right\rangle \right| \\
&= \lim_{t_1, t_2 \rightarrow 0^+} \left| \int_{\mathbb{R}^2} q_{t_1}(\gamma_1(u_1) - \gamma_1(v_1)) \varphi_1^{y_1, R_1}(v_1) \psi_1^{x_1, R_1}(u_1) \gamma_1'(v_1) \gamma_1'(u_1) dv_1 du_1 \right| \\
&\quad \times \left| \int_{\mathbb{R}^2} q_{t_2}(\gamma_2(u_2) - \gamma_2(v_2)) \varphi_2^{y_2, R_2}(v_2) \psi_2^{x_2, R_2}(u_2) \gamma_2'(v_2) \gamma_2'(u_2) dv_2 du_2 \right| \\
&\leq \lim_{t_1, t_2 \rightarrow 0^+} \int_{\mathbb{R}^2} |q_{t_1}(\gamma_1(u_1) - \gamma_1(v_1))| |\varphi_1^{y_1, R_1}(v_1) \psi_1^{x_1, R_1}(u_1) \gamma_1'(v_1) \gamma_1'(u_1)| dv_1 du_1 \\
&\quad \times \left| \int_{\mathbb{R}^2} \log((\gamma_2(u_2) - \gamma_2(v_2))^2) (\varphi_2^{y_2, R_2})'(v_2) \psi_2^{x_2, R_2}(u_2) \gamma_2'(u_2) dv_2 du_2 \right| \\
&= \lim_{t_1, t_2 \rightarrow 0^+} A_{t_1} \times B_{t_2}.
\end{aligned}$$

To estimate  $A_{t_1}$ , we use the kernel estimate for  $q_{t_1}$  to conclude the following bound.

$$\begin{aligned}
\int_{\mathbb{R}^2} |q_{t_1}(\gamma_1(u_1) - \gamma_1(v_1))| |\varphi_1^{y_1, R_1}(v_1) \psi_1^{x_1, R_1}(u_1) \gamma_1'(v_1) \gamma_1'(u_1)| dv_1 du_1 &\lesssim \int_{\mathbb{R}^2} \frac{1}{|u_1 - v_1|} |\varphi_1^{y_1, R_1}(v_1) \psi_1^{x_1, R_1}(u_1)| dv_1 du_1 \\
&\lesssim \frac{R_1^2}{|x_1 - y_1|} = \frac{R_1}{(R_1^{-1}|x_1 - y_1|)}.
\end{aligned}$$

For the second term, we argue exactly as in the full weak boundedness case using Lemma 5.1:

$$B_{t_2} \lesssim \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \log((\gamma_2(u_2) - \gamma_2(v_2))^2) R_2^{-1} (\varphi_2')^{y_2, R_2}(v_2) dv_2 \right| |\psi_2^{x_2, R_2}(u_2)| du_2 \lesssim \int_{\mathbb{R}} |\psi_2^{x_2, R_2}(u_2)| du_2 \lesssim R_2.$$

Therefore  $\tilde{C}_\Gamma$  satisfies (4.7). To prove (4.8), fix  $x_1, x_2, y_2 \in \mathbb{R}$ ,  $R_1, R_2 > 0$ , and  $\varphi_j, \psi_j$  for  $j = 1, 2$  as above, but furthermore assume (without loss of generality) that  $\gamma_1' \psi_1^{x_1, R_1}$  has mean zero. Since  $|x_1 - y_1| > 4R_1$

$$\begin{aligned}
& \left| \left\langle M_b \tilde{C}_\Gamma M_b(\varphi_1^{y_1, R_1} \otimes \varphi_2^{x_2, R_2}), \psi_1^{x_1, R_1} \otimes \psi_2^{x_2, R_2} \right\rangle \right| \\
&\leq \lim_{t_1, t_2 \rightarrow 0^+} \int_{\mathbb{R}^2} |q_{t_1}(\gamma_1(u_1) - \gamma_1(v_1)) - q_{t_1}(\gamma_1(x_1) - \gamma_1(v_1))| |\varphi_1^{y_1, R_1}(v_1) \psi_1^{x_1, R_1}(u_1) \gamma_1'(v_1) \gamma_1'(u_1)| dv_1 du_1 \\
&\quad \times \left| \int_{\mathbb{R}^2} \log((\gamma_2(u_2) - \gamma_2(v_2))^2) (\varphi_2^{y_2, R_2})'(v_2) \psi_2^{x_2, R_2}(u_2) \gamma_2'(u_2) dv_2 du_2 \right| \\
&= \lim_{t_1, t_2 \rightarrow 0^+} \tilde{A}_{t_1} \times B_{t_2}.
\end{aligned}$$

By the support properties of  $\varphi_1$  and  $\psi_1$ , we may assume that  $|y_1 - v_1| \leq R_1$  and  $|x_1 - u_1| \leq R_1$  to estimate the following part of the integrand from  $\tilde{A}_{t_1}$ :

$$\begin{aligned}
& |q_{t_1}(\gamma_1(u_1) - \gamma_1(v_1)) - q_{t_1}(\gamma_1(x_1) - \gamma_1(v_1))| \\
&= \left| \frac{(\gamma_1(u_1) - \gamma_1(v_1))(\gamma_1(x_1) - \gamma_1(v_1))^2 - (\gamma_1(x_1) - \gamma_1(v_1))(\gamma_1(u_1) - \gamma_1(v_1))^2}{[(\gamma_1(x_1) - \gamma_1(v_1))^2 + t_1^2][(\gamma_1(u_1) - \gamma_1(v_1))^2 + t_1^2]} \right. \\
&\quad \left. + \frac{(\gamma_1(u_1) - \gamma_1(v_1))t_1^2 - (\gamma_1(x_1) - \gamma_1(v_1))t_1^2}{[(\gamma_1(x_1) - \gamma_1(v_1))^2 + t_1^2][(\gamma_1(u_1) - \gamma_1(v_1))^2 + t_1^2]} \right| \\
&\leq \frac{|\gamma_1(u_1) - \gamma_1(v_1)| |\gamma_1(x_1) - \gamma_1(v_1)| |\gamma_1(x_1) - \gamma_1(u_1)|}{[(\gamma_1(u_1) - \gamma_1(v_1))^2 + t_1^2][(\gamma_1(x_1) - \gamma_1(v_1))^2 + t_1^2]} \\
&\quad + t_1^2 \frac{|\gamma_1(u_1) - \gamma_1(x_1)|}{|(\gamma_1(u_1) - \gamma_1(v_1))^2 + t_1^2| |(\gamma_1(x_1) - \gamma_1(v_1))^2 + t_1^2|} \\
&\lesssim \frac{|u_1 - v_1| |x_1 - v_1| |x_1 - u_1|}{|u_1 - v_1|^2 |x_1 - v_1|^2} + \frac{|x_1 - u_1|}{|x_1 - v_1|^2} \lesssim \frac{R_1}{|x_1 - y_1|^2}.
\end{aligned}$$

In the last line, we use that  $|x_1 - y_1| > R_1/4$ ,  $|x_1 - u_1| \leq R_1$ ,  $|y_1 - v_1| \leq R_1$ ,

$$|u_1 - v_1| \geq |x_1 - y_1|/2, \quad \text{and} \quad |x_1 - v_1| \geq |x_1 - y_1|/2.$$

It easily follows that

$$\tilde{A}_{t_1} \lesssim \frac{R_1}{|x_1 - y_1|^2} \int_{\mathbb{R}^2} |\phi_1^{y_1, R_1}(v_1) \psi_1^{x_1, R_1}(u_1)| dv_1 du_1 \lesssim \frac{R_1^3}{|x_1 - y_1|^2} = \frac{R_1}{(R_1^{-1}|x_1 - y_1|)^2},$$

as required in (4.8) with  $n_1 = \gamma = 1$ .

This verifies the first mixed weak boundedness properties (4.7) and (4.8) for  $\tilde{C}_\Gamma$ , and the other two conditions follow by symmetry.  $\square$

**Proposition 5.3.** *Assume  $\Gamma$  satisfies the hypotheses of Theorem 1. The operator  $\tilde{C}_\Gamma$  satisfies the  $Tb = T^*\tilde{b} = 0$  conditions with  $b(x) = \tilde{b}(x) = \gamma'_1(x_1)\gamma'_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^2$ .*

*Proof.* Let  $\eta_R \in C_0^\infty(\mathbb{R}^{n_1})$  be as above,  $\phi_1, \psi_1 \in C_0^\infty(\mathbb{R}^{n_1})$ , and  $\psi_2 \in C_0^\infty(\mathbb{R}^{n_2})$  such that  $\gamma'_1\psi_1$  and  $\gamma'_2\psi_2$  have mean zero. We use Proposition 5.1 to compute

$$\begin{aligned} \left\langle \tilde{C}_\Gamma(\gamma'_1\eta_R \otimes \gamma'_2\psi_2), \gamma'_1\psi_1 \otimes \gamma'_2\psi_2 \right\rangle &= \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \\ &\quad \times (\eta_R)'(y_1) \phi_2'(y_2) \psi_1(x_1) \psi_2(x_2) \gamma'_1(x_1) \gamma'_2(x_2) dy dx \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log((\gamma_1(x_1) - \gamma_1(Ry_1))^2) \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \eta'(y_1) \phi_2'(y_2) \psi_1(x_1) \psi_2(x_2) \gamma'_1(x_1) \gamma'_2(x_2) dy dx \\ &= \int_{\mathbb{R}^2} F_R(x_1) \left( \int_{\mathbb{R}} \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \phi_2'(y_2) dy_2 \right) \psi_1(x_1) \psi_2(x_2) \gamma'_1(x_1) \gamma'_2(x_2) dx, \\ &\quad \text{where } F_R(x_1) = \int_{\mathbb{R}} \log((\gamma_1(x_1) - \gamma_1(Ry_1))^2) \eta'(y_1) dy_1. \end{aligned}$$

Since  $\eta \in C_0^\infty(\mathbb{R})$ , it follows that  $\eta'$  has mean zero. Note also that  $Re(c_1) = 1$  since  $\gamma_1(x_1) = x_1 + iL_1(x_1)$  and  $L_1$  is real-valued, so  $\log(y_1^2 c_1^2)$  is well defined for  $y_1 \neq 0$ . Recall the definition of  $c_1$  in the hypotheses of Theorem 1. Hence we can also write  $F_R(x_1)$  in the following way.

$$F_R(x_1) = \int_{\mathbb{R}} [\log((\gamma_1(x_1) - \gamma_1(Ry_1))^2) - \log(R^2)] \eta'(y_1) dy_1 = \int_{\mathbb{R}} \log\left(\frac{(\gamma_1(x_1) - \gamma_1(Ry_1))^2}{R^2}\right) \eta'(y_1) dy_1.$$

Now we note that for all  $x_1 \in \mathbb{R}$  and  $y_1 \neq 0$

$$\lim_{R \rightarrow \infty} \log\left(\frac{(\gamma_1(x_1) - \gamma_1(Ry_1))^2}{R^2}\right) = \lim_{R \rightarrow \infty} \log\left(y_1^2 \frac{(\gamma_1(x_1) - \gamma_1(Ry_1))^2}{y_1^2 R^2}\right) = \log(y_1^2 c_1^2).$$

Recall that we have assumed  $\gamma_1(u_1)/u_1 \rightarrow c_1$  as  $|u_1| \rightarrow \infty$ . For  $R$  large enough so that  $\text{supp}(\psi_1) \subset B(0, R/2)$ , it follows that for  $x_1 \in \text{supp}(\psi_1)$  and  $y_1 \in \text{supp}(\eta') \subset B(0, 2) \setminus B(0, 1)$

$$\frac{|\gamma_1(x_1) - \gamma_1(Ry_1)|^2}{R^2} \geq (1 - \lambda_1^2) \frac{|x_1 - Ry_1|^2}{R^2} \geq (1 - \lambda_1^2) \frac{R^2 - |x_1|^2}{R^2} \geq 1 - \lambda_1^2.$$

We also have

$$\frac{|\gamma_1(x_1) - \gamma_1(Ry_1)|^2}{R^2} \leq \frac{4|x_1 - Ry_1|^2}{R^2} \leq \frac{4|x_1|^2}{R^2} + 4|y_1|^2 \leq 20$$

Therefore

$$\left| \log\left(\frac{(\gamma_1(x_1) - \gamma_1(Ry_1))^2}{R^2}\right) \eta'(y_1) \right| \lesssim \eta'(y_1).$$

Then by dominated convergence,

$$\lim_{R \rightarrow \infty} F_R(x_1) = \int_{\mathbb{R}} \log(y_1^2 c_1^2) \eta'(y_1) dy_1 = c.$$

Now  $F_R(x_1) \rightarrow c$  for some constant  $c \in \mathbb{C}$ , which does not depend on  $x_1$ . Since  $F_R(x_1)$  is bounded independent of  $x_1$ , we apply dominated convergence again to conclude

$$\begin{aligned} \lim_{R \rightarrow \infty} \langle \tilde{C}_\Gamma(\gamma'_1 \eta_R \otimes \gamma'_2 \varphi_2), \gamma'_1 \psi_1 \otimes \gamma'_2 \psi_2 \rangle &= \int_{\mathbb{R}^2} c \left( \int_{\mathbb{R}} \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \phi'_2(y_2) dy_2 \right) \psi_1(x_1) \psi_2(x_2) \gamma'_1(x_1) \gamma'_2(x_2) dx \\ &= c \left( \int_{\mathbb{R}} \psi_1(x_1) \gamma'_1(x_1) dx_1 \right) \left( \int_{\mathbb{R}^2} \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \phi'_2(y_2) \psi_2(x_2) \gamma'_2(x_2) dy_2 dx_2 \right) = 0. \end{aligned}$$

Here we use that  $\gamma'_1 \psi_1$  has mean zero. By symmetry, this holds when  $\gamma'_1 \phi_1$  has mean zero in place of  $\gamma'_1 \psi_1$ . Hence the  $\tilde{C}_\Gamma(b) = 0$  condition is satisfied, and the adjoint condition follows by symmetry.  $\square$

By Theorem 4, it follows that  $\tilde{C}_\Gamma$  can be extended to a bounded linear operator on  $L^p(\mathbb{R}^2)$  for  $1 < p < \infty$ . Hence  $C_\Gamma$  can be defined for  $g \in L^p(\Gamma)$  for  $1 < p < \infty$ , and for  $g \in L^p(\Gamma)$ , it follows that

$$\begin{aligned} \|C_\Gamma g\|_{L^p(\Gamma)}^p &= \int_{\mathbb{R}^2} |\tilde{C}_\Gamma M_b(g \circ \gamma)(x)|^p |\gamma'_1(x_1) \gamma'_2(x_2)| dx \\ &\leq \|\gamma'_1\|_{L^\infty} \|\gamma'_2\|_{L^\infty} \|\tilde{C}_\Gamma\|_{L^p, L^p}^p \int_{\mathbb{R}^2} |(g \circ \gamma)(x)|^p dx \\ &\leq 4 \|(\gamma'_1)^{-1}\|_{L^\infty} \|(\gamma'_2)^{-1}\|_{L^\infty} \|\tilde{C}_\Gamma\|_{L^p, L^p}^p \int_{\mathbb{R}^2} |g(x)|^p |\gamma'_1(x_1) \gamma'_2(x_2)| dx \leq 4 \|\tilde{C}_\Gamma\|_{L^p, L^p}^p \|g\|_{L^p(\Gamma)}^p. \end{aligned}$$

Furthermore for  $f \in C_0^\infty(\mathbb{R}^2)$ , there exists a constant  $C_{f,p} > 0$  such that

$$|\tilde{C}_t M_b f(x)|^p \leq C_{f,p} \left( \chi_{|x_1| \leq 2R_0} + \frac{1}{|x_1|^p} \chi_{|x_1| > 2R_0} \right) \left( \chi_{|x_2| \leq 2R_0} + \frac{1}{|x_2|^p} \chi_{|x_2| > 2R_0} \right),$$

where  $R_0$  is large enough so that  $\text{supp}(f) \subset B(0, R_0/2)$ . Then by dominated convergence, it follows that

$$\lim_{t_1, t_2 \rightarrow 0^+} \tilde{C}_t M_b f = \tilde{C}_\Gamma M_b f \quad \text{in } L^p(\mathbb{R}^2).$$

One can argue by density to verify that  $\tilde{C}_\Gamma$  extends to all of  $L^p(\mathbb{R}^2)$  and that  $\tilde{C}_t f \rightarrow \tilde{C}_\Gamma f$  in  $L^p(\mathbb{R}^2)$  for  $f \in L^p(\mathbb{R}^2)$  as  $t_1, t_2 \rightarrow 0^+$  for all  $1 < p < \infty$ .

It easily follows that for  $g \in L^p(\Gamma)$  where  $1 < p < \infty$

$$\lim_{t_1, t_2 \rightarrow 0^+} C_t g = C_\Gamma g$$

in  $L^p(\Gamma)$ . We prove now Theorem 3, in order to conclude about the almost everywhere convergence  $C_t g(z) = C g(z)$  as  $t_1, t_2 \rightarrow 0^+$ . We need the following lemma.

**Lemma 5.2.** *Let  $z_j, \xi_j \in \Gamma_j, j = 1, 2$ . Then, the following relationship holds*

$$\int_{\Gamma_j} q_{t_j}(z - \xi) p_{s_j}(\xi - \zeta) d\xi = q_{t_j + s_j}(z - \zeta)$$

for every  $t_j, s_j \neq 0$ .

*Proof.* It is not hard to prove that the conclusion follows using residue theorems similar to the proof of Lemma 2.1.  $\square$

Finally, we prove Theorem 3.

*Proof.* Fix  $p \in (1, \infty)$  and suppose for the moment that  $C_t g = C_{t_1, t_2} g = P_{t_1, t_2} C_\Gamma g$  holds whenever  $g$  is a function in  $L^p(\Gamma)$  and  $t_1, t_2 \neq 0$ . Then, using some of the estimates we used in the proof of Lemma 2.1, it is not hard to prove that

$$C_\Gamma^* g(\gamma(x)) = \sup_{t_1, t_2 > 0} |P_t C_\Gamma g \circ \gamma(x)| \leq \mathcal{M}_S(C_\Gamma g \circ \gamma)(x),$$

where  $\mathcal{M}_S$  is the biparameter strong maximal operator. Thus, the  $L^p$  boundedness of the maximal operator  $C_\Gamma^*$  follows from the boundedness of the operators  $\mathcal{M}_S$  and  $C_\Gamma$ .

It remains to prove the equality  $C_t g = P_t C_\Gamma g$ . Without losing generality, we can suppose that  $g \circ \gamma$  is in  $C_0^\infty(\mathbb{R}^2)$ , so that the existence of the pointwise limit  $C_t g(z)$  is guaranteed by Proposition 5.1. Thus, using Lemma 5.2, we obtain



$$\begin{aligned}
C_I g(z) &= \frac{1}{\pi^2} \int_{\Gamma_1 \times \Gamma_2} \frac{z_1 - \xi_1}{(z_1 - \xi_1)^2 + t_1^2} \frac{z_2 - \xi_2}{(z_2 - \xi_2)^2 + t_2^2} g(\xi) d\xi \\
&= \lim_{s_1, s_2 \rightarrow 0^+} \frac{1}{\pi^2} \int_{\Gamma_1 \times \Gamma_2} \frac{z_1 - \xi_1}{(z_1 - \xi_1)^2 + (s_1 + t_1)^2} \frac{z_2 - \xi_2}{(z_2 - \xi_2)^2 + (s_2 + t_2)^2} g(\xi) d\xi \\
&= \lim_{s_1, s_2 \rightarrow 0^+} \frac{1}{\pi^4} \int_{(\Gamma_1 \times \Gamma_2)^2} \frac{t_1}{(z_1 - \zeta_1)^2 + t_1^2} \frac{\zeta_1 - \xi_1}{(\zeta_1 - \xi_1)^2 + s_1^2} \frac{t_2}{(z_2 - \zeta_2)^2 + t_2^2} \frac{\zeta_2 - \xi_2}{(\zeta_2 - \xi_2)^2 + s_2^2} d\zeta g(\xi) d\xi \\
&= \lim_{s_1, s_2 \rightarrow 0^+} \frac{1}{\pi^2} \int_{\Gamma_1 \times \Gamma_2} \frac{t_1}{(z_1 - \zeta_1)^2 + t_1^2} \frac{t_2}{(z_2 - \zeta_2)^2 + t_2^2} \left( \frac{1}{\pi^2} \int_{\Gamma_1 \times \Gamma_2} \frac{\zeta_1 - \xi_1}{(\zeta_1 - \xi_1)^2 + s_1^2} \frac{\zeta_2 - \xi_2}{(\zeta_2 - \xi_2)^2 + s_2^2} g(\xi) d\xi \right) d\zeta \\
&= \frac{1}{\pi^2} \int_{\Gamma_1 \times \Gamma_2} \frac{t_1}{(z_1 - \zeta_1)^2 + t_1^2} \frac{t_2}{(z_2 - \zeta_2)^2 + t_2^2} C_\Gamma g(\zeta) d\zeta = P_{t_1, t_2} C_\Gamma g(z),
\end{aligned}$$

Therefore, the  $L^p$  boundedness of the maximal operator  $C_\Gamma^*$  is proved. The almost everywhere pointwise convergence for a general function  $g$  in  $L^p(\Gamma)$  can be now obtained using the existence of the pointwise limit for smooth functions, the boundedness of  $C_\Gamma^*$ , and a standard argument. See, for example, [Gra04].  $\square$

This completes the proof of the first part of Theorem 2, pertaining to  $C_\Gamma$ .

## 6. BOUNDS FOR $C_\Gamma^{p1}$ , $C_\Gamma^{p2}$ , $\tilde{C}_\Gamma^{p1}$ , AND $\tilde{C}_\Gamma^{p2}$

Like in the last section, we define the parameterized versions of  $C_\Gamma^{p1}$  and  $C_\Gamma^{p2}$ , for  $f \in C_0^\infty(\mathbb{R}^2)$  and  $x \in \mathbb{R}^2$

$$\begin{aligned}
\tilde{C}_\Gamma^{p1} M_b f(x) &= \lim_{t_1, t_2 \rightarrow 0^+} \tilde{C}_\Gamma^{p1} M_b f(x), \quad \text{where} \quad \tilde{C}_\Gamma^{p1} M_b f(x) = \int_{\mathbb{R}^2} q_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) p_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) f(y) b(y) dy, \\
\tilde{C}_\Gamma^{p2} M_b f(x) &= \lim_{t_1, t_2 \rightarrow 0^+} \tilde{C}_\Gamma^{p2} M_b f(x), \quad \text{where} \quad \tilde{C}_\Gamma^{p2} M_b f(x) = \int_{\mathbb{R}^2} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) q_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) f(y) b(y) dy.
\end{aligned}$$

We prove these bounds by applying the single parameter  $Tb$  theorem from [DJS85]. We outline the proof that  $\tilde{C}_\Gamma^{p1}$  and  $\tilde{C}_\Gamma^{p2}$  are bounded on  $L^p(\Gamma)$ . The details can be deciphered from the previous more complicated biparameter versions. Define for  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{C}$  and  $x_1, x_2 \in \mathbb{R}$

$$\begin{aligned}
\tilde{C}_{\Gamma_1} M_{\gamma_1'} f_1(x_1) &= \lim_{t_1 \rightarrow 0^+} \int_{\mathbb{R}} q_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) f_1(y_1) \gamma_1'(y_1) dy_1, \\
\tilde{C}_{\Gamma_2} M_{\gamma_2'} f_2(x_2) &= \lim_{t_2 \rightarrow 0^+} \int_{\mathbb{R}} q_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) f_2(y_2) \gamma_2'(y_2) dy_2.
\end{aligned}$$

The following propositions are routine given the proofs of Propositions 5.1, 5.2, and 5.3.

**Proposition 6.1.** *Assume  $\Gamma$  satisfies the hypotheses of Theorem 1. For all  $f \in C_0^\infty(\mathbb{R}^2)$  and  $x \in \mathbb{R}^2$ ,*

$$\begin{aligned}
\tilde{C}_\Gamma^{p1}(bf)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \partial_{y_1} f(y_1, x_2) dy_1, \\
\tilde{C}_\Gamma^{p2}(bf)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \partial_{y_2} f(x_1, y_2) dy_2.
\end{aligned}$$

Also, for all  $f, g \in C_0^\infty(\mathbb{R}^2)$ , the pairings  $\langle \tilde{C}_\Gamma^{p1}(bf), bg \rangle$  and  $\langle \tilde{C}_\Gamma^{p2}(bf), bg \rangle$  can be realized as any of the following absolutely convergent integrals:

$$\begin{aligned}
\langle \tilde{C}_\Gamma^{p1}(bf), bg \rangle &= \frac{1}{2\pi} \int_{\mathbb{R}^3} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) \partial_{y_1} f(y_1, x_2) g(x) b(x) dy_1 dx, \\
\langle \tilde{C}_\Gamma^{p1}(bf), bg \rangle &= -\frac{1}{2\pi} \int_{\mathbb{R}^3} \log((\gamma_1(x_1) - \gamma_1(y_1))^2) f(y_1, x_2) \partial_{x_1} g(x) b(y_1, x_2) dy_1 dx,
\end{aligned}$$

$$\begin{aligned}\langle \tilde{C}_\Gamma^{p2}(bf), bg \rangle &= \frac{1}{2\pi} \int_{\mathbb{R}^3} \log((\gamma_2(x_2) - \gamma_2(y_2))^2) \partial_{y_2} f(x_1, y_2) g(x) b(x) dy_2 dx, \\ \langle \tilde{C}_\Gamma^{p2}(bf), bg \rangle &= -\frac{1}{2\pi} \int_{\mathbb{R}^3} \log((\gamma_2(x_2) - \gamma_2(y_2))^2) f(x_1, y_2) \partial_{x_2} g(x) b(x_1, y_2) dy_2 dx.\end{aligned}$$

**Proposition 6.2.** Assume  $\Gamma$  satisfies the hypotheses of Theorem 1. The operator  $\tilde{C}_{\Gamma_1}$  and  $\tilde{C}_{\Gamma_2}$  satisfies the single parameter weak boundedness property.

**Proposition 6.3.** Assume  $\Gamma$  satisfies the hypotheses of Theorem 1. The operator  $\tilde{C}_{\Gamma_1}$  and  $\tilde{C}_{\Gamma_2}$  satisfies the cancellation conditions  $\tilde{C}_{\Gamma_1}(\gamma'_1) = \tilde{C}_{\Gamma_1}^*(\gamma'_1) = \tilde{C}_{\Gamma_2}(\gamma'_2) = \tilde{C}_{\Gamma_2}^*(\gamma'_2) = 0$ .

Then by the  $Tb$  theorem of David-Journé-Semmes [DJS85], it follows that  $\tilde{C}_{\Gamma_1}$  and  $\tilde{C}_{\Gamma_2}$  are bounded on  $L^p(\mathbb{R})$ . It follows that for  $f, g \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}\left| \langle \tilde{C}_\Gamma^{p1}(bf), bg \rangle \right| &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \lim_{t_1 \rightarrow 0^+} \int_{\mathbb{R}^2} \log((\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2) \partial_{y_1} f(y_1, x_2) g(x) \gamma'_1(x_1) dy_1 dx_1 \right| |\gamma'_2(x_2)| dx_2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \lim_{t_1 \rightarrow 0^+} \int_{\mathbb{R}^2} q_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) f(y_1, x_2) \gamma'_1(y_1) g(x) \gamma'_1(x_1) dy_1 dx_1 \right| |\gamma'_2(x_2)| dx_2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \langle \tilde{C}_{\Gamma_1}(\gamma'_1 f(\cdot, x_2)), \gamma'_1 g(\cdot, x_2) \rangle \right| |\gamma'_2(x_2)| dx_2 \\ &\lesssim \int_{\mathbb{R}} \|f(\cdot, x_2)\|_{L^p(\mathbb{R})} \|g(\cdot, x_2)\|_{L^{p'}(\mathbb{R})} dx_2 \leq \|f\|_{L^p(\mathbb{R}^2)} \|g\|_{L^{p'}(\mathbb{R}^2)}.\end{aligned}$$

Therefore  $\tilde{C}_\Gamma^{p1}$  is bounded on  $L^p(\mathbb{R}^2)$  for  $1 < p < \infty$ , and by symmetry  $\tilde{C}_\Gamma^{p2}$  is as well. Again it follows that for  $f \in L^p(\mathbb{R}^2)$

$$\lim_{t_1, t_2 \rightarrow 0^+} \tilde{C}_t^{p1} M_b f = \tilde{C}_{\Gamma_1} M_b f \quad \text{and} \quad \lim_{t_1, t_2 \rightarrow 0^+} \tilde{C}_t^{p2} M_b f = \tilde{C}_{\Gamma_2} M_b f \quad \text{in } L^p(\mathbb{R}^2),$$

and for  $g \in L^p(\Gamma)$

$$\lim_{t_1, t_2 \rightarrow 0^+} C_t^{p1} g = C_{\Gamma_1}^{p1} g \quad \text{and} \quad \lim_{t_1, t_2 \rightarrow 0^+} C_t^{p2} g = C_{\Gamma_2}^{p2} g \quad \text{in } L^p(\Gamma).$$

This completes the proof for the  $L^p$  convergence. Similarly, the almost everywhere convergence can be derived with arguments analogous to the ones used in the biparameter situation. We prefer not to report detail again since it is clear by now how to proceed.

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